Lec 2/28

Tuesday, February 28, 2017 09:08

more fixed point stuff

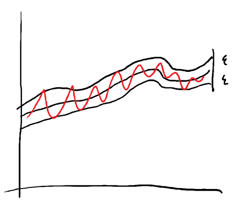
Corollaries of Banach Contraction Mapping theorem:

① $\rho(x_n, x) \le \frac{\alpha^n}{1-\alpha} \rho(x_0, f(x_0)) = \frac{\alpha^n}{1-\alpha} c$ a priori estimate

② $\rho(x_n, x) \le \alpha \rho(x_{n-1}, x)$

③ $\rho(x_n, x) \le \frac{\alpha}{1-\alpha} \rho(x_{n-1}, x_n)$ a posteriori estimate

$C[0,1]$ = Cont. functions on $[0,1]$. $\rho(f,g) = \max_{x \in [0,1]} |f(x) - g(x)|$



— epsilon-close to —

$C^1[0,1]$: continuously diffable functions on $[0,1]$.

$\rho(f,g) = \max_{x \in [0,1]} |f(x) - g(x)| + \max_{x \in [0,1]} |f'(x) - g'(x)|$ exercise: show this is a metric, make a picture (like the one above)

$X = \{0,1\}^{\mathbb{N}}$ = all mappings from $\mathbb{N}$ to $\{0,1\}$ ≡ all sequences of 0s and 1s.

$t \in (0,1)$, $t = \sum_{i=1}^{\infty} \frac{b_i}{2^i}$ where $b_i \in \{0,1\}$ (binary expansion).

There is natural 1-1 correspondence btwn $X = \{0,1\}^{\mathbb{N}}$ and $P(\mathbb{N})$.

for each sequence of 0s and 1s, if it is 1 it is in a subset.

$S \subseteq \mathbb{N} \longleftrightarrow 1_S(n) = \begin{cases} 1 & n \in S \\ 0 & n \notin S \end{cases}$ 1_S is characteristic function of a set S

$S \subseteq \mathbb{N} \longleftrightarrow \underset{\{0,1\}^{\mathbb{N}}}{\overset{}{1_S}}(n) = \begin{cases} 1 & n \in S \\ 0 & n \notin S \end{cases}$ $\quad 1_S$ is characteristic function of a set S.

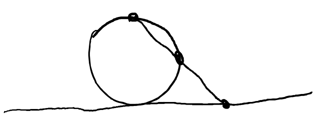
($1_S \in \{0,1\}^{\mathbb{N}}$)

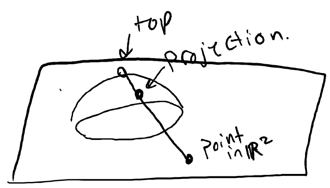
Define a metric ρ on $X = \{0,1\}^{\mathbb{N}}$ as follows: $\rho((x_n),(y_n)) = \sum_{i=1}^{\infty} \frac{|x_n - y_n|}{2^n}$

Exercise 1: ρ is a metric

Exercise 2: (X, ρ) is compact

$\mathbb{R}^2 \approx \mathbb{C}$





Stereographic projection.

• used in mapmaking

exercise: this mapping preserves angles between curves. (conformal mapping).

Theorem (Weierstrass) for any $f \in C[0,1]$ and any ε, $\exists P(x) \in \mathbb{R}(x)$ (finite polynomials w/ real coeffs.)

s.t. $\rho(P, f) = \max_{x \in [0,1]} |P(x) - f(x)| < \varepsilon$.

(this means in metric space $C[0,1]$, the set $\mathbb{R}[x]$ is dense)

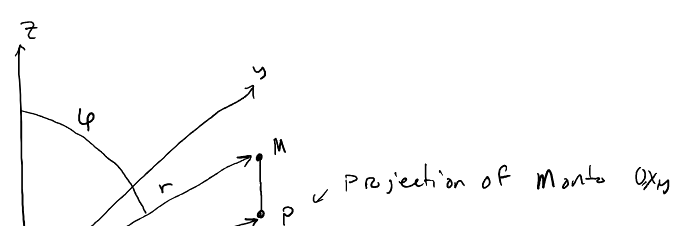
Def A metric space (X,ρ) is separable if $\exists$ dense countable subset in (X,ρ)

Exercise: $(C[0,1], \rho)$ is separable (use $\mathbb{Q}[x]$ to approx $\mathbb{R}[x]$)

Spherical Coordinates:

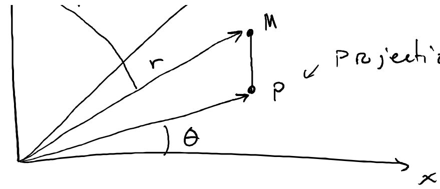
$x = r \sin\varphi \cos\theta$

$y = r \sin\varphi \sin\theta$



$y = r \sin\varphi \sin\theta$

$z = r\cos\varphi$



$0 \le r < \infty$

$0 \le \varphi \le \pi$

$0 \le \theta < 2\pi$

$$J = \underset{3\times3}{\left| \quad \right|} = r^2 \sin\varphi$$ (extra exercise)

Def a set $S \subseteq \mathbb{R}$ has measure 0 if $\forall \varepsilon$ it can be covered by a collection of intervals w/ total length $< \varepsilon$. exercise: show it doesn't matter whether open or closed.

any finite set has measure 0. any countable set is measure 0.

$S = \{x_1, x_2, \dots\}$ so choose an interval of length $\frac{\varepsilon}{2^i}$ around x_i.