Lec 2/24

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Theorem: Banach Contraction Principle

If $f: X \to X$ s.t. $\rho(f(x), f(y)) \leq \alpha \rho(x, y)$ $\forall x, y \in X$ (complete)

then $\exists$ a unique fixed pt x s.t $\forall x_0 \in X$, $f^{(n)}(x_0) \to X$. ← n applications

Problem: $\mathbb{Q} \cap [0,1]$. prove $\lim_{n\to\infty} x_n = \frac{p}{q}$ leads to contradiction.

Liner Algebra:

$A = (a_{ij})_{i,j=1}^{n}$ $\qquad Ax = b$ is a system of eqns. If $\det(A) \neq 0$ There is a solution: $A^{-1}b$.

A^{-1} exists $\Updownarrow$ $\det(A) \neq 0$

$Ax = b \Leftrightarrow 0 = -Ax + b \Leftrightarrow x = (I - A)x + b$. let $Tx := (I - A)x + b$.

Think about $Tx = x$. $\qquad T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$

we have such a fixed pt if T is a contraction map.

Distance: define $\rho(x, x') = \max_j |x_j - x_j'|$ $\qquad$ want to think about $\rho(Tx, Tx')$

Condition (assumption)

If $I - A = \alpha_{ij}$ $\quad (= -a_{ij} + \delta_{ij})$ ← kronecker symbol, $= 1$ if $i = j$ 0 otherwise.

(*) $\sum_{j=1}^{n} |\alpha_{ij}| < K < 1$ $\quad \forall i = 1, \ldots, n$ $\qquad$ Prove this also implies A invertible

$$\rho(Tx, Tx') = \max_i \left| \sum_{i=1}^{n} \alpha_{ij}(x_j - x_j') \right| \leq \max_i \sum_{j=1}^{n} |\alpha_{ij}| |x_j - x_j'| \leq \max_i \left(\sum_{j=1}^{n} |\alpha_{ij}| \right) \cdot \max_j |x_j - x_j'|$$

$$< K \rho(x, x')$$

so can apply Banach Contraction Principle

• Key: rewrite equation so $x = f(x)$ for some transformation f.

$x(t) = f(t) + \int^{t} k(t,s)\, x(s)\, ds$ ← Volterra equation

$$y(t) = f(t) + \int_a^t \underbrace{K(t,s)\, y(s)\, ds}_{Ay} \quad \Leftarrow \text{ Voltera equation}$$

equivalent

$\Updownarrow$ $y = f + Ay \longrightarrow$ find fixed points which are functions.

$y \in C[a,b]$, $a \le t \le b$.

<u>Def</u>: $C[a,b]$ is the space of continuous functions $[a,b] \to \mathbb{R}$.

with distance $\rho(f,g) = \max_{x \in [a,b]} |f(x) - g(x)|$ Exercise: show ρ is a metric on $C[0,1]$ for convenience

Exercise: $d(f,g) = \int_a^b |f-g|$ is not a complete metric on $C[a,b]$.