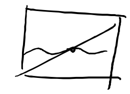
Lec 2/23

Thursday, February 23, 2017 09:09

Fixed point theorems

object invariant under transformation.

Theorem for any cts $f:[0,1]\to[0,1]$ $\exists x_0\in[0,1]$ s.t. $f(x_0)=x_0$



Theorem If $f:[0,1]^2\to[0,1]^2$ is cts $\exists (x_0,y_0)\in[0,1]^2$ s.t. $f(x_0,y_0)=(x_0,y_0)$

Theorem: Brouwer's Fixed point theorem:

if f is a continuous map of a closed ball in $\mathbb{R}^n$ into itself, then $\exists$ a fixed point $f(\vec{x})=\vec{x}$.

Counterexample. not every cts map of a circle into itself has fixed points

rotate by a little bit.

What happens with sphere in higher dimensions?

Complete metric spaces: & Banach Contraction principle

Let (X,ρ) be a complete metric space and let $f:X\to X$ satisfy for some $\alpha\in(0,1)$ $\rho(f(x),f(y))\le\alpha\rho(x,y)$. Then $\exists$ a unique fixed point $x\in X$. moreover, for any $x_0\in X$, the iterates x_0, $f(x_0)=x_1$, $f(x_1)=x_2$, ... converges to x.

(every cauchy sequence has a limit in X).

Prove: $\mathbb{Q}\cap[0,1]$ is not complete.

come up with cauchy sequence which does not converge.

(infinite continued fraction might not converge).

Proof: (Uniqueness assuming existence). If $f(x)=x$, $f(y)=y$ then

$$\rho(x,y)\le\alpha\rho(f(x),f(y))=\alpha\rho(x,y)\Rightarrow\rho(x,y)=0\Rightarrow x=y.$$

(existence). Let x_0 be any point in X, $x_{n+1} = f(x_n)$, $n = 0, 1, 2, \dots$

We have: $\rho(x_n, x_{n+1}) = \rho(f(x_{n-1}), f(x_n)) \leq \alpha \rho(x_{n-1}, x_n) \cdots \leq \alpha^n \underbrace{\rho(x_0, x_1)}_{c}$.

$\rho(x_n, x_{n+k}) \leq \sum_{j=1}^{k} \rho(x_{n+j-1}, x_{n+j}) \leq \left(\sum_{j=1}^{\infty} \alpha^{n+j-1}\right) c \leq \frac{c\alpha^n}{1-\alpha} \Rightarrow \{x_n\}$ is cauchy.

Then $x_n \to x$ for some $x \in X$ (completeness).

Since f is cts, $f(x_n) \to f(x)$

$$\begin{array}{ccc} f(x_n) & \to & f(x) \\ \| & & \| \\ x_{n+1} & \to & x \end{array}$$

note: any contraction map is continuous

Extra Exercises:

$X = \mathbb{R}$: $f(x) = \sqrt{x^2+1}$, $f(x) = \log(1+e^{-x})$, $f(x) = x + \frac{1}{x}$ $\Bigg\}$ $|f(x) - f(x')| < |x - x'|$ $\forall x \neq x'$ on X.

- Check that there is no fixed point for each example.

However if (X, ρ) is compact and $f: X \to X$ satisfies $\rho(f(x), f(y)) < \rho(x, y)$ $\forall x \neq y$ then $\exists$ fixed point!!!! (maybe unique).

- Prove this.