Lec 2/17

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Spectral theorem for Quadratic Forms.

A symmetric $n \times n$ matrix.

$Q_A : \mathbb{R}^n \to \mathbb{R}$, $Q_A(\vec{x}) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$

Theorem (spectral theorem): There are n mutually perpendicular unit vectors $\vec{u}_1, \ldots, \vec{u}_n$ such that $Q_A(\vec{x}) = \sum_{i=1}^{n} \lambda_i (\vec{x} \cdot \vec{u}_i)^2$ and λ_i are eigenvalues of A, given by solving $cp(\lambda) = \det(A - \lambda I) = 0$ for λ and $\vec{u}_i$ are eigenvectors so $Au_i = \lambda_i u_i$.

Notation: $\vec{x} \cdot \vec{y} = \underset{1 \times n}{\vec{x}^t} \overset{\text{matrix multiplication}}{*} \underset{n \times 1}{\vec{y}}$

C^t is transpose of C = interchange rows w/ columns.

$(C * D)^t = D^t * C^t$

$B_A(\vec{x}, \vec{y}) = \sum_{i=1}^{n} a_{ij} x_i y_j$

$Q_A(\vec{x}) = \vec{x} \cdot A\vec{x} = \vec{x}^t * A * \vec{x}$

B_A bilinear product $B_A : \mathbb{R}^{2n} \to \mathbb{R}$, $B_A(\vec{x}, \vec{y}) = \vec{x} \cdot A\vec{y} = \vec{x}^t * A * \vec{y}$

Note $B_A(\vec{y}, \vec{x}) = \vec{y}^t * A * \vec{x} = \vec{y}^t * A^t * \vec{x} = (\vec{x}^t * A * \vec{y})^t = \vec{x} \cdot A\vec{y} = B_A(\vec{x}, \vec{y})$.

Definition V is a linear subspace of $\mathbb{R}^n$ if it is a nonempty subset of $\mathbb{R}^n$ and if $\forall \vec{x}, \vec{y} \in V$, $\vec{x} + \vec{y} \in V$ and $\forall \alpha \in \mathbb{R}$, $\alpha\vec{x} \in V$

Note these imply $\vec{0} \in V$. Note linear subspaces are closed subsets:

If W subset of $\mathbb{R}^n$, $W^\perp = \bigcap_{\vec{w} \in W} \underbrace{\{\vec{v} \in \mathbb{R}^n : \vec{v} \cdot \vec{w} = 0\}}_{\text{closed}}$ and $V^{\perp\perp} = V$.

Main Lemma: Let V be a linear subspace of $\mathbb{R}^n$ such that $A(V) \subseteq V$. Let $\vec{u} \in \overbrace{V \cap S^{n-1}}^{\text{compact since } V \text{ closed}}$ s.t. $Q_A(\vec{u})$ is maximal over $V \cap S^{n-1}$. Then

(a) $\vec{w} \in V$ & $\vec{w} \cdot \vec{u} = 0 \Rightarrow \vec{u} \cdot A\vec{w} = 0$

(b) $A\vec{u} = \lambda\vec{u}$ for some $\lambda \in \mathbb{R}$.

Proof of lemma: for any θ, take $\vec{v} = \frac{\vec{w}}{|\vec{w}|}$. $\cos(\theta)\vec{u} + \sin(\theta)\vec{v} \in V \cap S^{n-1}$. This is because

$$(\cos(\theta)\vec{u} + \sin(\theta)\vec{v})\cdot(\cos(\theta)\vec{u} + \sin(\theta)\vec{v}) = \underset{1}{\underset{\downarrow}{\cos^2(\theta)|\vec{u}|^2}} + 2\cos(\theta)\sin(\theta)\underset{0}{\underset{\downarrow}{|\vec{u}||\vec{v}|}} + \sin^2(\theta)\underset{1}{\underset{\downarrow}{|\vec{v}|^2}}$$

$$= \cos^2(\theta) + \sin^2(\theta) = 1.$$

$$f(\theta) \underset{}{\overset{Q_A(\cos(\theta)\vec{u},\ \sin(\theta)\vec{v})}{=}} (\cos(\theta)\vec{u} + \sin(\theta)\vec{v})\cdot A(\cos(\theta)\vec{u} + \sin(\theta)\vec{v})$$ has a maximum at $\theta = 0$.

$$= \cos^2(\theta)\,\vec{u}\cdot A\vec{u} + \sin^2(\theta)\,\vec{v}\cdot A\vec{v} + \cos(\theta)\sin(\theta)\,\vec{u}\cdot A\vec{v} + \sin(\theta)\cos(\theta)\,\vec{v}\cdot A\vec{u}.$$

$$= \cos^2(\theta)\,\vec{u}\cdot A\vec{u} + \sin^2(\theta)\,\vec{v}\cdot A\vec{v} + \underbrace{2\cos(\theta)\sin(\theta)}_{\sin(2\theta)}\,\vec{u}\cdot A\vec{v}$$

maximum

$$f'(\theta) = -2\cos(\theta)\sin(\theta)\,\vec{u}\cdot A\vec{u} + 2\sin(\theta)\cos(\theta)\,\vec{v}\cdot A\vec{v} + 2\cos(2\theta)\,\vec{u}\cdot A\vec{v}.$$

$$0 = f'(0) = 0 + 0 + 2\vec{u}\cdot A\vec{v} \Rightarrow \vec{u}\cdot A\vec{v} = 0.$$ that proves pt (a).

Now $A\vec{u} \in V$ and so can be expressed in the form $\lambda\vec{u} + \vec{w}$ where $\vec{w} \perp \vec{u}$.

$$A\vec{u} = \lambda\vec{u} + \vec{w}$$

$$\vec{u}\cdot A\vec{u} = \lambda|\vec{u}|^2 + \vec{u}\cdot\vec{w} = \lambda, \text{ so } \vec{w} = A\vec{u} - (\vec{u}\cdot A\vec{u})\vec{u}$$

$$\underset{\underset{0}{\parallel}}{\underset{\vec{u}\cdot A\vec{w}}{\underset{\parallel}{\vec{w}\cdot A\vec{u}}}} = \lambda\underset{\underset{0}{\parallel}}{\vec{u}\cdot\vec{w}} + |\vec{w}|^2 \Rightarrow \vec{w} = \vec{0} \text{ so } A\vec{u} = \lambda\vec{u}.$$

This proves part (b). ∎

Proof of spectral theorem:

Inductively Define a decreasing sequence of linear subspaces $V_1 \supseteq V_2 \supseteq V_3 \supseteq \dots \supseteq V_n$ and elements $\vec{u}_i \in V_i$ satisfying hypothesis of main lemma; $A(V_i) \subseteq V_i$ and $Q_A(\vec{u}_i)$ is maximal on $V_i \cap S^{n-1}$.

Start by defining $V_1 = \mathbb{R}^n$, $Q_A(\vec{u}_1)$ maximal on S^{n-1}. Having defined V_i,

let $V_{i+1} = \{\vec{v} \in V_i : \vec{v}\cdot\vec{u}_i = 0\} = \{\vec{u}_i\}^{\perp} \cap V_i$.

then by part (a) of the main lemma, we have $A(V_{i+1}) \subseteq V_{i+1}$.

Let $Q_A(u_{i+1})$ be maximal on $V_{i+1} \cap S^{n-1}$. $\dim V_{i+1} = \dim V_i - 1$.

$\Rightarrow \dim V_i = n+1-i. \Rightarrow \dim V_n = 1.$

Then $\{\vec{u}_1, \dots, \vec{u}_n\}$ are mutually perpendicular unit vectors so they form a basis of $\mathbb{R}^n$. Also by part (b) of main lemma, we have $Au_i = \lambda_i u_i$ for $i = 1, \dots, n$

So if $\vec{x} \in \mathbb{R}^n$ then $\vec{x} = \sum_{i=1}^{n} \alpha_i u_i \Rightarrow \vec{u}_j \cdot \vec{x} = \alpha_j$.

$$Q_A(\vec{x}) = \vec{x} \cdot A\vec{x} = \left(\sum_{i=1}^{n} (\vec{u}_i \cdot \vec{x}) \vec{u}_i\right) \cdot A \left(\sum_{i=1}^{n} (\vec{u}_i \cdot \vec{x}) \, \vec{u}_i\right)$$

$$= \left(\sum_{i=1}^{n} (\vec{u}_i \cdot \vec{x}) \vec{u}_i\right) \cdot \left(\sum_{j=1}^{n} (\vec{u}_j \cdot \vec{x}) \lambda_j \vec{u}_j\right)$$

$$= \sum_{i,j=1}^{n} (\vec{u}_i \cdot \vec{x})(\vec{u}_j \cdot \vec{x}) \, \lambda_j (\vec{u}_i \cdot \vec{u}_j)$$

$$= \sum_{j=1}^{n} \lambda_j \, (\vec{u}_j \cdot \vec{x})$$