Lec 2/10

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Find the maximum/minimum values of $f(x,y,z) = 2x^3 + 2y^2 + 3z^2$ on unit sphere

$g(x,y,z) = x^2+y^2+z^2 = 0$ compact set.

Lagrangian: $L_{f,g}(x,y,z,\lambda) = f(x,y,z) - \lambda g(x,y,z).$

$= 2x^3 + 2y^2 + 3z^2 - \lambda(x^2+y^2+z^2-1)$

find critical points of Lagrangian:

1) $0 = \frac{\partial L}{\partial \lambda} = x^2+y^2+z^2-1$

2) $0 = \frac{\partial L}{\partial x} = 6x^2 - 2\lambda x = 2x(3x-\lambda)$

3) $0 = \frac{\partial L}{\partial y} = 4y - 2\lambda y = 2y(2-\lambda)$

4) $0 = \frac{\partial L}{\partial z} = 6z - 2\lambda z = 2z(3-\lambda)$

Case 1: $x=0$. then at least one of y & z is nonzero.

Case 1a: $y \neq 0$. then $2-\lambda = 0 \Rightarrow \lambda = 2$.

so $3-\lambda \neq 0$ so $z=0$. also $y^2=1 \Rightarrow y=\pm 1$

So we get $(0,\pm 1, 0, 2)$ are crit pts.

Case 1b: $z \neq 0$ then $3-\lambda = 0 \Rightarrow \lambda = 3$ so $2-\lambda \neq 0$ so $y=0$ so $z=\pm 1$

$(0,0,\pm 1, 3)$ crit pts

Case 2: $x \neq 0$ so $3x-\lambda = 0$ so $\lambda = 3x$.

so $0 = 2y(2-3x)$

$0 = 2z(3-3x)$

Case 2a: $y=0$, $z=0$. so $x=\pm 1$ so $\lambda = \pm 3$

$\pm(1,0,0,3)$ crit pts

Case 2b: $y \neq 0$ so $x = 2/3 \Rightarrow z=0 \Rightarrow (\frac{2}{3})^2 + y^2 = 0 \Rightarrow y = \pm\frac{\sqrt{5}}{3}$

so crit pts: $(\frac{2}{3}, \frac{\pm\sqrt{5}}{3}, 0, 2)$

Case 2C: $z \neq 0$ so $x = 1 \Rightarrow 1 + (z^2) > 1$ so no points.

Critical pts (x,y,z)	$f(x,y,z) = 2x^3 + 2y^2 + 3z^2$
$(0,0,\pm 1)$	3
$(0,\pm 1,0)$	2
$(1,0,0)$	2
$(-1,0,0)$	-2
$\left(\frac{2}{3}, \frac{\pm\sqrt{5}}{3}, 0\right)$	$\frac{16}{27} + \frac{10}{9} = \frac{46}{27} < 2$

So the function is maxed to 3 at $(0,0,\pm 1)$. mined to -2 at $(-1,0,0)$.

- We derived lagrange multiplier method by noting that at a local max or min $\vec{a}$ satisfying the constraint we have that both $\nabla f(\vec{a})$ and $\nabla g(\vec{a})$ perpendicular to tangent plane to $g(\vec{x})=0$ at $\vec{a}$.

$\Rightarrow$ $\nabla f(\vec{a})$ and $\nabla g(\vec{a})$ must be parallel. This only works if $\nabla g(\vec{a}) \neq \vec{0}$

* if there are points $g(\vec{a})=0$ and $\nabla g(\vec{a}) = \vec{0}$, there may be a local max/min there.

- Lagrange multiplier method can be extended to more than one constraint: Optimize $f(\vec{x})$ ($\mathbb{R}^n$) subject to k constraints $g_i(\vec{x}) = 0$ $i = 1, \dots, k$

Then $L(\underbrace{\vec{x}, \lambda_1, \dots, \lambda_k}_{n+k \text{ vars}}) = f(\vec{x}) - \sum_{j=1}^{k} \lambda_j g_j(\vec{x})$

Critical points of L are candidates for max/min of f subj to constraints.

↳ all of $\nabla f(\vec{a}), \nabla g_1(\vec{a}), \dots, \nabla g_k(\vec{a})$ are perpendicular to any tangent vector to curve $S = \{\vec{x} : g_i(\vec{x}) = 0 \ \forall i = 1, \dots, k\}$ which passes through $\vec{a}$.

$n-k$ dimensional submanifold

assuming $\{\nabla g_j(\vec{a})\}$ are linearly independent, it follows that

$$\nabla f(\vec{a}) = \sum_{j=1}^{k} \lambda_j g_j(\vec{a})$$