Lec 2/1

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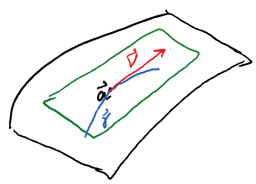
Proposition: Suppose that a hypersurface in $\mathbb{R}^{n+1}$ is given by $F(\vec{x}) = 0$. ← a set S

If $\vec{a} \in S$, then the equation of hyperplane tangent to S at $\vec{a}$ is

$$\nabla F(\vec{a}) \cdot (\vec{x} - \vec{a}).$$

Informal proof:



given a vector $\vec{v}$ in tangent hyperplane we can find a smooth curve

$\vec{\gamma} : (-\varepsilon, \varepsilon) \to S \subseteq \mathbb{R}^{n+1}$ with

$\vec{\gamma}(0) = a$, $\vec{\gamma}'(0) = \vec{v}$

Then consider the composite

$$\underbrace{(-\varepsilon, \varepsilon) \xrightarrow{\vec{\gamma}} S \subseteq \overset{\text{open}}{U} \subseteq \mathbb{R}^{n+1} \xrightarrow{F} \mathbb{R}}_{\longrightarrow \text{ constant } 0 \text{ function}}$$

$$\frac{d(F \circ \vec{\gamma})}{dt}(t) = 0$$

$\parallel$ chain rule

$$0 = DF(\vec{a})\, D\vec{\gamma}(0) = \nabla f(\vec{a}) \cdot \underbrace{\vec{v}}_{= \vec{\gamma}'(0)}$$

So $\nabla F(\vec{a})$ is perpendicular to $\vec{v}$ arbitrary vector in tangent hyperplane

$\Rightarrow \nabla F(\vec{a})$ is normal to hyperplane.

Eqn of hyperplane w/ normal vector $\nabla F(\vec{a})$ and passing thru $\vec{a}$ is:

$$\nabla F(\vec{a}) \cdot (\vec{x} - \vec{a}) = 0 \qquad (*)$$

Compare this w/ previous form:

Surface $S = \{ z = f(\vec{x}) \}$ $\qquad f : U \subseteq \mathbb{R}^n \to \mathbb{R}$

$= \{ f(\vec{x}) - z = 0 \}$

Suppose $\vec{b} \in U$. Then $(\vec{b}, f(\vec{b})) \in S$

Call $F(\vec{x}, z) = f(\vec{x}) - z$

eqn of tangent hyperplane given by $(*)$ is

$$\nabla F(\vec{b}, f(\vec{b})) \cdot \left((\vec{x}, z) - (\vec{b}, f(\vec{b}))\right) = 0$$

observe: $\partial_i F = \partial_i f$ for $i = 1, \dots, n$

$\partial_{n+1} F = -1$

so $\nabla F(\vec{b}, f(\vec{b})) = (\partial_1 f, \partial_2 f, \dots, \partial_n f, -1)$

$= \nabla f, -1$

$$(\nabla f(\vec{b}), -1)\,(\vec{x} - \vec{b}, z - f(\vec{b})) = 0$$

$$\nabla f(\vec{b}) \cdot (\vec{x} - \vec{b}) - z + f(\vec{b}) = 0$$

$$z = f(\vec{b}) + \nabla f(\vec{b}) \cdot (\vec{x} - \vec{b})$$

2.1 #6: Suppose f is 3 times differentiable on an open interval containing a

1) $\displaystyle \lim_{h \to 0} \frac{f(a+2h) - 2f(a+h) + f(a)}{h^2} = f''(a)$

2) $\displaystyle \lim_{h \to 0} \frac{f(a+3h) - 3f(a+2h) + 3f(a+h) - f(a)}{h^3} = f^{(3)}(a)$

Suggested Solution: Apply L'H repeatedly:

1) $\displaystyle \lim_{h \to 0} \frac{2f'(a+2h) - 2f'(a+h)}{2h} = \lim_{h \to 0} \frac{2f''(a+2h) - f''(a+h)}{1} \overset{f'' \text{ continuous}}{=} 2f''(a) - f''(a) = f''(a)$

2) 3-fold application gives $\displaystyle \lim_{h \to 0} \frac{9f^{(3)}(a+3h) - 8f^{(3)}(a+2h) + f^{(3)}(a+h)}{2}$

assuming $f^{(3)}(x)$ is continuous at $x = a$ gives this equal to $f^{(3)}(a)$

open $\mathbb{R}^2$

assuming $f'(x)$ is continuous at $x=a$ gives this eq. ...

Theorem: Let $f: \underset{\text{open}}{U} \subseteq \mathbb{R}^2 \to \mathbb{R}$ and $(a,b) \in U$, and suppose that $\partial_1 f$, $\partial_2 f$, and $\partial_2 \partial_1 f$ are defined on U and $\partial_2 \partial_1 f$ is continuous at (a,b). Then $\partial_1 \partial_2 f\,(a,b)$ is defined and equal to $\partial_2 \partial_1 f\,(a,b)$.

Proof:
$$\partial_2 \partial_1 f(a,b) = \lim_{y \to b} \frac{\partial_1 f(a,y) - \partial_1 f(a,b)}{y-b}$$

$$= \lim_{y \to b} \frac{\lim_{x \to a} \frac{f(x,y) - f(a,y)}{x-a} - \lim_{x \to a} \frac{f(x,b) - f(a,b)}{x-a}}{y-b}$$

$$= \lim_{y \to b} \lim_{x \to a} \left.\frac{f(x,y) - f(a,y) - f(x,b) + f(a,b)}{(x-a)(y-b)} \right\} F(x,y)$$

$F(x,y)$ defined on $\overbrace{B_\infty(r,(a,b)) \cap \{y \neq b, x \neq a\}}^{S}$ for some $r > 0$

Strategy: show that a stronger version of this limit exists:

i.e. show $\displaystyle\lim_{(x,y) \to (a,b),\ (x,y) \in S} F(x,y)$ exists.

$\forall (x,y) \in S$ & $\forall\, \varepsilon > 0\ \exists\, \underset{\hat{r}}{\delta} > 0$ s.t $0 < |x-a| < \delta,\ 0 < |y-b| < \delta \Rightarrow |F(x,y) - L| < \varepsilon$

$$\lim_{y \to b} \lim_{x \to a} F(x,y) = \partial_2 \partial_1 f(a,b)$$

$\Leftrightarrow \forall \varepsilon > 0\ \exists\, \delta > 0,\ \delta_y > 0$ s.t

$0 < |x-a| < \delta_y,\quad 0 < |y-b| < \delta \Rightarrow |F(x,y) - \partial_2 \partial_1 f(a,b)| < \varepsilon$

Define $\varphi(x) = f(x,y) - f(x,b)$, $(x,y) \in B_\infty(r,(a,b))$

$$F(x,y) = \frac{\varphi(x) - \varphi(a)}{(x-a)(y-b)} \overset{\text{MVT on } x}{=} \frac{\varphi'(x_1)\cancel{(x-a)}}{\cancel{(x-a)}(y-b)}$$ where x_1 btwn x and a

and $\varphi'(x_1) = \partial_1 f(x_1,y) - \partial_1 f(x_1,b)$

$$\overset{\text{MVT on } y}{=} \partial_2 \partial_1 f(x_1, y_1)$$ where y_1 btwn y and b

hence $F(x,y) = \partial_2 \partial_1 f(x_1, y_1)$

$$|F(x,y) - \partial_2 \partial_1 f(a,b)| = |\partial_2 \partial_1 f(x_1,y_1) - \partial_2 \partial_1 (a,b)| < \varepsilon$$

$\cap \ (x \neq a, y \neq b)$

Provided $(x,y) \in B_\infty(\delta, (a,b))$ for some δ.

because $\partial_2 \partial_1 f$ is continuous.

so part 1 is proved.

stronger limit exists

$$\text{Now } \partial_1 \partial_2 f(a,b) = \lim_{x \to a} \lim_{y \to b} F(x,y) = \lim_{y \to b} \lim_{x \to a} F(x,y) = \partial_2 \partial_1 f(a,b).$$