Lec 1/9

Sunday, January 8, 2017 23:35

Limits & Cont.

$f: \mathbb{R} \to \mathbb{R}$ is cts at $a \in \mathbb{R}$ if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

$f: \mathbb{R}^2 \to \mathbb{R}$ is cts at (a,b) if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |f(x,y) - f(a,b)| < \varepsilon$

both use distance measures

General Case:

$$f: (X, d_1) \to (Y, d_2)$$

X, Y metric spaces w/ distances d_1, d_2.

$f: X \to Y$ is cts at $a \in X$ if $\forall \varepsilon > 0 \ \exists \delta$ s.t. $d_1(x, a) < \delta \Rightarrow d_2(f(x), f(a)) < \varepsilon$

Metric space: set X and function $d: X \times X \to [0, \infty)$ s.t.

1) $d(x,y) = 0$ iff $x = y$

2) $d(x,y) = d(y,x)$

3) $d(x,y) \le d(x,z) + d(y,z)$ (triangle inequality)

(if ↗ is = then z is 'between' x and y)

Alternative definition for cts in $\mathbb{R}^2 \to \mathbb{R}$:

$f: \mathbb{R}^2 \to \mathbb{R}$ is cts at (a,b) if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\underbrace{|x-a| < \delta \ \& \ |y-b| < \delta}_{\Updownarrow} \Rightarrow |f(x,y) - f(a,b)| < \varepsilon$

This definition corresponds to the <u>box metric</u>. $\max(|x-a|, |y-b|) < \delta$

$d_\infty: \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ (box metric in plane)

$d_\infty((x_1, y_1), (x_2, y_2)) = \max(|x_1 - x_2|, |y_1 - y_2|)$

Clearly it satisfies (1) and (2).

triangle ineq:

$|x_1 - x_3| \le |x_1 - x_2| + |x_2 - x_3| \le \max(|x_1 - x_2|, |y_1 - y_2|) + \max(|x_2 - x_3|, |y_2 - y_3|)$

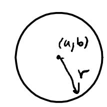
$$|x_1 - x_3| \le |x_1 - x_2| + |x_2 - x_3| \le \max(|x_1 - x_2|, |y_1 - y_2|) + \max(|x_2 - x_3|, |y_2 - y_3|)$$

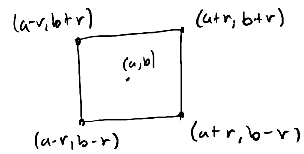
$$= d_\infty((x_1, y_1), (x_2, y_2)) + d_\infty((x_2, y_2), (x_3, y_3))$$

so is $|y_1 - y_3|$

so $d_\infty((x_1, y_1), (x_3, y_3)) \le$

Geometry of metrics d_∞, d_2 (euclidean) on $\mathbb{R}^2$

$\{(x,y) \mid d_2((x,y), (a,b)) < r\} =$ 

$\{(x,y) \mid d_\infty((x,y), (a,b)) < r\} =$ 

(both exclude boundary)

Is notion of cts the same under d_2 and d_∞? yes.

Lemma: (1) $id : (\mathbb{R}^2, d_2) \to (\mathbb{R}^2, d_\infty)$
(2) $id : (\mathbb{R}^2, d_\infty) \to (\mathbb{R}^2, d_2)$

are continuous everywhere.

Proof: (1) Given $\varepsilon > 0$, want to find $\delta > 0$ s.t. $\sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow \max(|x-a|, |y-b|) < \varepsilon$

$\delta = \varepsilon$ will work

(2) Given $\varepsilon > 0$, wtf $\delta > 0$ s.t. $\max(|x-a|, |y-b|) < \delta \Rightarrow \sqrt{(x-a)^2 + (y-b)^2} < \varepsilon$

$\delta = \frac{\varepsilon}{\sqrt{2}}$ will work.

Most useful metrics on $\mathbb{R}^n$ arise from a simpler notion: norm = distance from origin.

Definition A norm on $\mathbb{R}^n$ is a function $\nu : \mathbb{R}^n \to [0, \infty)$ s.t.

a) $\nu(\vec{x}) = 0$ iff $\vec{x} = \vec{0}$

b) $\nu(\alpha \vec{x}) = |\alpha| \nu(\vec{x})$ where $\alpha \in \mathbb{R}$

c) $\nu(\vec{x} + \vec{y}) \le \nu(\vec{x}) + \nu(\vec{y})$

Examples:

euclid $\nu_2(\vec{x}) = \sqrt{\sum_{i=1}^{n} x_i^2}$

euclid $\quad \nu_2(\vec{x}) \quad \sqrt{\sum_{i=1} x_i}$

box $\quad \nu_\infty(\vec{x}) = \max\{|x_i|\}_{i=1}^{n}$

taxicab $\quad \nu_1(\vec{x}) = \sum_{i=1}^{n}|x_i|$

Given a norm $\nu:\mathbb{R}^n \to [0,\infty)$, can define a distance $d_\nu(\vec{x},\vec{y}) = \nu(\vec{x}-\vec{y})$

$(a) \Rightarrow (1)$: $d_\nu(\vec{x},\vec{y}) = 0$ when $\vec{x}-\vec{y} = \vec{0}$

$(b) \Rightarrow (2)$: $\alpha = -1 \Rightarrow d_\nu(\vec{x},\vec{y}) = d_\nu(\vec{y},\vec{x})$

$(c) \Rightarrow (3)$: $d_\nu(\vec{x},\vec{z}) \leq d_\nu(\vec{x},\vec{y}) + d_\nu(\vec{y},\vec{z})$

$= \nu((\vec{x}-\vec{y})+(\vec{y}-\vec{z})) \quad \ldots$

Shorthand: $d_{\nu_p}(\vec{x},\vec{y}) = d_p(\vec{x},\vec{y}) \qquad p = 1, 2, \infty$

there is an infinite family of norms (hence metrics) on $\mathbb{R}^n$

$\nu_p : \mathbb{R}^n \to [0,\infty) \qquad p \in [1,\infty)$

$\nu_p(\vec{x}) = \sqrt[p]{\sum_{i=1}^{n} x_i^p}$

$\lim_{p\to\infty} \nu_p(\vec{x}) = \nu_\infty(\vec{x})$

Problem on take-home final:
Prove ν_p satisfies triangle inequality.

taxicab norm/distance in $\mathbb{R}^2$ is a good approx to distance btwn 2 pts in a city whose streets are in a rectangular grid.