Lec 1/27

Friday, January 27, 2017 09:12

Eqn of tangent plane at (x_0, y_0):

$$z = f(x_0, y_0) + \partial_1 f(x_0, y_0)(x - x_0) + \partial_2 f(x_0, y_0)(y - y_0)$$

Condition: $$\lim_{(x,y)\to(x_0,y_0)} \frac{|f(x,y) - T(x,y)|}{|(x,y) - (x_0,y_0)|} = 0.$$

<u>Example</u>: $f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}$, $(x,y) \neq (0,0)$; 0, o.w. Claim: there is no tangent plane at $(0,0,0)$

$$f(x,0) = 0 \Rightarrow \partial_1 f(0,0) = 0$$

$$f(y,0) = 0 \Rightarrow \partial_2 f(0,0) = 0$$

if there is a tangent plane, it is $z = 0$.

but $$\lim_{(x,y)\to(0,0)} \frac{\left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right|}{\sqrt{x^2+y^2}} = 0 \ ?$$

$$= \lim_{(x,y)\to(0,0)} \frac{|xy|}{x^2+y^2}$$

approach along the line $\vec{\gamma}(t) = (t,t)$

$$= \lim_{t\to 0} \frac{t^2}{2t^2} \quad \frac{1}{2} \neq 0.$$

<u>Problem</u>: find tangent hyperplane to the graph $z = f(\vec{x})$ where $f: \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}$ (graph $\subseteq \mathbb{R}^{n+1}$)

at a given point $(\vec{x}_0, f(\vec{x}_0))$

Eqn of hyperplane:

$$z = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n + c$$

$$= \vec{a} \cdot \vec{x} + c \quad \text{where } \vec{a} = (a_1, \ldots, a_n)$$

Since hyperplane must pass thru $(\vec{x}_0, f(\vec{x}_0))$, we must have $f(\vec{x}_0) = \vec{a}\cdot\vec{x}_0 + c$

so $c = f(\vec{x}_0) - \vec{a}\cdot\vec{x}_0$

so $z = \vec{a}\cdot\vec{x} + f(\vec{x}_0) - \vec{a}\cdot\vec{x}_0$

$= \vec{a}\cdot(\vec{x}-\vec{x}_0) + f(\vec{x}_0) = T(\vec{x})$.

Definition: Let $U \subseteq \mathbb{R}^n$ be an open set and $f: U \to \mathbb{R}$.

f is differentiable at $\vec{x}_0 \in U$ if $\exists \vec{a} \in \mathbb{R}^n$ so that

$$\lim_{\vec{x}\to\vec{x}_0} \frac{|f(\vec{x}) - f(\vec{x}_0) - \vec{a}(\vec{x}-\vec{x}_0)|}{|\vec{x}-\vec{x}_0|} = 0. \quad (\ast)$$

($\operatorname{dom} f = U$ open $\Rightarrow$ limit is strong limit).

Remark: it does not matter which norm is used on $\mathbb{R}^n$.

By comparability of norms,

$A\nu(\vec{x}-\vec{x}_0) \le |\vec{x}-\vec{x}_0| \le B\nu(\vec{x}-\vec{x}_0)$ for some $A, B \in \mathbb{R}^+$

So the limit would be adjusted by multiplication of a constant and would still be 0. (also some squeeze theorem involved).

a convenient reformulation of $(\ast)$. let $\vec{h} = \vec{x}-\vec{x}_0 \Rightarrow \vec{x} = \vec{x}_0 + \vec{h}$.

$$(\ast) \Leftrightarrow \lim_{\vec{h}\to 0} \frac{|f(\vec{x}_0+\vec{h}) - f(\vec{x}_0) - \vec{a}\cdot\vec{h}|}{|\vec{h}|} = 0.$$

Theorem: if f is differentiable at $\vec{x}_0$ then $\vec{a} = (\partial_1 f(\vec{x}_0), \partial_2 f(\vec{x}_0), \dots, \partial_n f(\vec{x}_0))$

$= \vec{\nabla} f(\vec{x}_0)$.

Proof: let $\vec{h} = (0, 0, \dots, \overset{\text{i-th coordinate}}{h}, \dots, 0, 0)$.

Then $(*) \Leftrightarrow \lim_{h\to 0} \left| \dfrac{f(x_{01},\dots,x_{0i}+h,\dots,x_{0n}) - f(\vec{x}_0) - h a_i}{h} \right| = 0$

$$= \left| \lim_{h\to 0} \frac{f(x_{01},\dots,x_{0i}+h,\dots,x_{0n}) - f(\vec{x}_0) - h a_i}{h} \right|$$

$$= \left| \partial_i f(\vec{x}_0) - a_i \right| = 0 \Rightarrow a_i = \partial_i f(\vec{x}_0) \qquad \blacksquare$$

Condition $(*)$ is difficult to check, but there is a stronger condition which is easier to check:

Theorem: Suppose $U \subseteq \mathbb{R}^n$ is open, and all partial derivatives $\partial_i f : U \to \mathbb{R}$ are defined and continuous for $i = 1, 2, \dots, n$.
Then f is differentiable on U.

Proof: Let $\vec{h} = (h_1, \dots, h_n)$ be given. Let $\vec{h}^{(i)} = (0, \dots, 0, h_i, h_{i+1}, \dots, h_n)$ for $i = 0, \dots, n$.
We define $\vec{h}^{(n+1)} = \vec{0}$. By the single-variable mean value theorem,

$$\underbrace{f(\vec{x}_0 + \vec{h}^{(i)}) - f(\vec{x}_0 + \vec{h}^{(i+1)})}_{\text{all components except } i\text{-th are equal!}} = \partial_i f(\vec{c}^{(i)}) h_i$$

$$\pi_i(\vec{x}_0 + \vec{h}^{(i)}) = x_{0i} + h_i, \quad \pi_i(\vec{x}_0 + \vec{h}^{(i+1)}) = x_{0i}$$

where $\vec{c}_i = (x_{01}, x_{02}, \dots, x_{0i-1}, \underset{x_{0i}}{\overset{x_{0i}+h_i}{\underset{\vee|}{\overset{\vee|}{c_i}}}}, x_{0i+1} + h_{i+1}, \dots, x_{0n} + h_n)$

note $f(\vec{x}_0 + \vec{h}) - f(\vec{x}_0) = \sum_{i=1}^{n} \left[f(\vec{x}_0 + \vec{h}^{(i)}) - f(\vec{x}_0 + \vec{h}^{(i+1)}) \right]$ (telescoping series).

$$= \sum_{i=1}^{n} \left(\partial_i f(\vec{c}^{(i)}) h_i \right)$$

Now we have

$$\frac{\left| f(\vec{x}_0 + \vec{h}) - f(\vec{x}_0) - \sum_{i=1}^{n} \partial_i f(x_0) h_i \right|}{|\vec{h}|} \qquad (**)$$

$$= \frac{\left| \sum_{i=1}^{n} \left[\partial_i f(\vec{c}^{(i)}) - \partial_i f(x_0) \right] \right| |h_i|}{|\vec{h}|} \leq \sum_{i=1}^{n} \left| \partial_i f(\vec{c}^{(i)}) - \partial_i f(x_0) \right| \frac{|h_i|}{|\vec{h}|}$$

note that $\frac{|h_i|}{|\vec{h}|} = \frac{|h_i|}{\left| \sum_{j=1}^{n} h_j^2 \right|} \leq \frac{|h_i|}{\sqrt{h_i^2}} = 1$

$$\leq \sum_{i=1}^{n} \left| \partial_i f(\vec{c}^{(i)}) - \partial_i f(x_0) \right|.$$

as $\vec{h} \to \vec{0}$, $\vec{c}^{(i)} \to \vec{x}_0$

$$\Rightarrow \lim_{\vec{h} \to \vec{0}} \sum_{i=1}^{n} \left| \partial_i f(\vec{c}^{(i)}) - \partial_i f(x_0) \right| = 0,$$

So $\lim_{\vec{h} \to \vec{0}} (**) = 0$ by Sq. thm.