Lec 1/26

Thursday, January 26, 2017 09:16

Example: let X be the following subset of $\mathbb{R}^2$.

$$X = X_0 \cup X_1$$

$$X_0 = \{(0, y) : y \in [-1, 1]\}$$

$$X_1 = \{(x, \cos(\tfrac{1}{x})) : x \in (0, 1]\}$$

Show that X is connected but not path connected

1) X is connected.

Proof by contradiction: Suppose there is a separation $X = U \sqcup V$.

U, V open in X.

X_1 is connected: continuous image of $(0, 1]$.

$$\Rightarrow X_1 \subseteq U \Rightarrow \underset{\substack{\parallel \\ X}}{\overline{X_1}} \subseteq \overline{U} = U \Rightarrow V = \emptyset$$

2) X is not path connected.

let $\vec{P}_n = (\frac{1}{n}, (-1)^n) = (\frac{1}{n}, \cos(\pi \frac{1}{n})) \in X_1 \subseteq X$

Proof by contradiction. let $\vec{f} : [0, 1] \to X$ from $(0,0)$ to $\vec{P}_1 = (1, -1)$

We construct a decreasing sequence by induction

$$0 < \cdots < t_2 < t_1 \leq 1 \quad \text{such that} \quad \vec{f}(t_n) = P_n.$$

$U = X \cap \{(x,y) : x < \frac{1}{n}\}$
$V = X \cap \{(x,y) : x > \frac{1}{n}\}$

for $n = 1$, take $t_1 = 1$.

having constructed $t_1, \ldots, t_{n-1}$, want to find $\hat{t}_n < t_{n-1}$ s.t. $\vec{f}(t_n) = \vec{p}_n$

Suppose t_n does not exist. Then $\underbrace{\vec{f}([0, t_{n-1}])}_{\text{connected}} \subseteq X \setminus \{P_n\} = U \sqcup V$

so either $\vec{f}([0, t_{n-1}]) \subseteq U$ or $\vec{f}([0, t_{n-1}]) \subseteq V$. both are impossible since $\vec{f}(0) = (0,0) \in U$ and $\vec{f}(t_{n-1}) = \vec{p}_{n-1} \in V$.

So there is such a point t_n.

$\{t_n\}_{n=1}^{\infty}$ is a bounded decreasing sequence so it has a limit $u = \lim_{n\to\infty} t_n \in [0,1]$.

and since $\vec{f}$ is continuous, $\lim_{n\to\infty} \vec{f}(t_n) = f(u)$. but $\{\vec{f}(t_n)\}$ has no limit,

since it is $-1, 1, -1, 1, \ldots$ approaches $(0,-1)$ by one subseq, $(0,1)$ by another.

What is the right notion of derivative for multivariable functions?

Motivational problem of 1-var Differential calculus:

what is the equation of the tangent line to the graph of a function at some point?

Analogous problem for functions of 2 variables:

tangent plane $z = f(x,y)$ at a point $(x_0, y_0, f(x_0,y_0))$.

Eqn of tangent plane has the formula $z = Ax + By + C$.

must pass through the point $(x_0, y_0, f(x_0,y_0))$

so $f(x_0,y_0) = Ax_0 + By_0 + C \Rightarrow C = f(x_0,y_0) - Ax_0 - By_0$

so $$z = Ax + By + f(x_0,y_0) - Ax_0 - By_0$$
$$= f(x_0,y_0) + A(x-x_0) + B(y-y_0)$$

If we take the intersection of the graph $z = f(x,y)$ by the plane $y = y_0$, then $z = f(x, y_0)$ function of a single variable

and the intersection of the tangent plane is $z = f(x_0,y_0) + A(x-x_0) + 0$

which should be the equation of the tangent line to the graph of $z = f(x,y_0)$

Conclusion: $A = \left.\frac{dz}{dx}\right|_{x=x_0}$ hence $A = \lim_{x\to x_0} \frac{f(x,y_0) - f(x_0,y_0)}{x - x_0}$

or $$\lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h} = \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \partial_1 f(x_0,y_0)$$ (streamlined notation)

Similarly, $B = \partial_2 f(x_0, y_0)$

Conclusion: If there is a tangent plane to $z = f(x,y)$ at (x_0, y_0), then it has the form

$$z = f(x_0,y_0) + \left(\partial_1 f(x_0,y_0)\right)(x-x_0) + \left(\partial_2 f(x_0,y_0)\right)(y-y_0)$$

This is a necessary condition, but not a sufficient one.

In 1-var case.

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} \Rightarrow 0 = \lim_{x\to x_0}\left[\frac{f(x)-f(x_0)}{x-x_0} - f'(x_0)\right]$$

$$= \lim_{x\to x_0}\left[\frac{f(x)-f(x_0)-f'(x_0)(x-x_0)}{x-x_0}\right]$$

Note: $T(x) = f(x_0)+f'(x_0)(x-x_0)$ is eqn of tangent line.

$$= \lim_{x\to x_0}\left[\frac{|f(x)-T(x)|}{|x-x_0|}\right]$$

$|f(x)-T(x)|$ = distance in $\mathbb{R}$ btwn $f(x)$ and $T(x)$.

$|x-x_0|$ = " " x and x_0

Obvious generalization: (call eqn of tangent plane $T(x,y)$).

$$0 = \lim_{(x,y)\to(x_0,y_0)} \frac{|f(x,y)-T(x,y)|}{|(x,y)-(x_0,y_0)|}$$

distance in $\mathbb{R}^2$.