Lec 1/25

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Connectedness.

Definition A separation of a toporogical space (X, τ) is a pair of nonempty open sets (U, V) such that $X = U \cup V$ and $U \cap V = \emptyset$.
We say X is disconneeted if there is a separation and connected if there is no separation.

Remarks: $U = X \setminus V$ is closed, so is V (in X). we could also say "closed" in Definition

A subset $A \subseteq X$ is <u>clopen</u> if it is both closed & open.

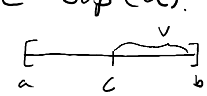
if $A \neq \emptyset$, $A \neq X$, then $X = A \cup (X \setminus A)$.

X conneeted $\Leftrightarrow$ the only clopen sets are $\emptyset$ and X.

Definition Let $A \subseteq (X, \tau)$ (X not necessarily connected). A is a conneeted subset of X if $(A, \tau_A = \{U \cap A : U \in \tau\})$ is connected.

$\Leftrightarrow$ If $A \subseteq U \cup V$, U, V open in X, $A \cap U \cap V = \emptyset$.
Then either $A \cap U = \emptyset$ or $A \cap V = \emptyset$

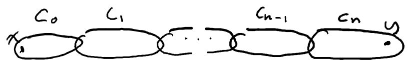
Theorem if $[a, b]$ is a closed finite interval, then $[a,b]$ is connected.

Proof: By contradiction. Suppose $[a,b] = U \cup V$ is a separation (U, V open in $[a,b]$)
We may suppose that $b \in V$. let $c = \sup(U)$. then $c \in \overline{U} = U$ since U is closed. $c < b$ because  . There are points in V arbitrarily close to c. so $c \in \partial V \subseteq \overline{V} = V$. so $\{c\} \in U \cap V = \emptyset$. ■

Lemma Suppose A is a connected subset of a disconnected space X, suppose $X = U \cup V$ is a separation. then either $A \subseteq U$ or $A \subseteq V$.

Proof: Otherwise $A = (U \cap A) \sqcup (V \cap A)$ is a separation. ■

Definition Let $(X, \mathcal{T})$ be a topological space, $x \neq y \in X$. A connectivity chain from x to y in X is a finite sequence $\{C_i\}_{i=0}^{n}$ of connected subsets such that $x \in C_0$, $y \in C_n$, $C_{i-1} \cap C_i \neq \emptyset$.



Theorem X is connected $\Leftrightarrow$ $\forall x, y \in X$, $\exists$ a connected chain between them.

Proof $\Rightarrow$ X is connected so take $\{X\}$ as connectivity chain

$\Leftarrow$ By contradiction. Suppose X is disc. then (U, V) disconnection. Pick $x \in U$, $y \in V$. by lemma, $C_0 \subseteq U$. but now (by induction) $C_i \subseteq U$ so $C_n \subseteq U$ so $y \in U$ but $U \cap V = \emptyset$.

∎

Corollary the only connected subsets of $\mathbb{R}$ are:

1) $\emptyset$
2) $\{c\}$
3) any kind of interval. (open, closed, finite, infinite, etc.).

Proof 1) is trivial. 2) any disc. set needs atleast 2 pts.

3) $[a, b]$ already done

$[a, \infty) = \bigcup_{n=ceil(a)+1}^{\infty} [a, n]$

$(a, \infty) = \bigcup_{n=ciel(a)+2} [a + \frac{1}{n}, n]$

these are connectivity chains.

etc.

conversely, if A is not one of these things, can find $c \in \mathbb{R} \setminus A$ s.t. $(-\infty, c) \cap A \neq \emptyset$ and $(c, \infty) \cap A \neq \emptyset$. so $(A \cap (-\infty, c), A \cap (c, \infty))$ is a separation.

Generalization of Intermediate Value theorem.

Theorem: Suppose $f: (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is continuous and onto. Suppose X is connected. then Y is connected.

Proof: by contradiction. Suppose Y is not connected. Then

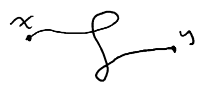
There is a separation $Y = U \sqcup V$. It follows that $X = f^{-1}(U) \sqcup f^{-1}(V)$ is a separation of X.

Corollary If $f: (X, \tau_1) \to (Y, \tau_2)$ is continuous & X is connected, so is $f(X)$.

Proof: f restricted to $f| : (X, \tau_1) \to (f(X), \tau_{2, f(X)})$

Corollary If $f: (X, \tau) \to \mathbb{R}$ is continuous, X is connected, then $f(X)$ is either a point or an interval.

Definition A (normalized) path from x to y in a topological space (X, τ) is a continuous function $f: [0,1] \to X$ s.t. $f(0) = x$, $f(1) = y$



A space is path connected if there is a path between any two points

Theorem If (X, τ) is path connected, it is connected.

Proof take $\{f([0,1])\}$ as the connectivity chain between x and y. where f is a normalized path btwn x and y.