Lec 1/24

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Generalizations of Extreme Value Theorem.

Theorem: Let $f: X \to Y$ be continuous & onto. If X is compact, so is Y.

Proof: Let $\{V_\alpha\}_{\alpha \in \mathcal{Q}}$ be an open cover of Y. Then $\{f^{-1}(V_\alpha)\}_{\alpha \in \mathcal{Q}}$ is an open cover of X since $X = f^{-1}(Y) = \bigcup_{\alpha \in \mathcal{Q}} f^{-1}(V_\alpha)$. Since X is compact, there is a finite subcover $\{f^{-1}(V_{\alpha_i})\}_{i=1}^{n}$.

$$Y \underset{\text{onto}}{=} f(X) = f\left(\bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})\right) = \bigcup_{i=1}^{n} f(f^{-1}(V_{\alpha_i}))$$

$$\text{onto} \rightarrow = \bigcup_{i=1}^{n} V_{\alpha_i}$$

■

Reformulation Theorem: Let $f: X \to Y$ be cts, X compact. Then $f(X)$ is a compact subset of Y.

Corollary: (Extreme Value theorem for $\mathbb{R}^n$): If $A \subseteq \mathbb{R}^n$ is a compact subset, and $f: A \to \mathbb{R}$ is continuous everywhere, then f takes on both a maximum and a minimum.

Proof $f(A)$ is a closed interval (a compact subset of $\mathbb{R}$) ■

The following results hold for metric spaces (not general topological spaces)

Theorem Bolzano-Weierstrass property. Let (X, d) be a compact metric space. Then any sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ has a convergent subsequence in X.

Proof: By contradiction. Suppose $\{x_n\}_{n=1}^{\infty}$ has no convergent subsequence then for any $z \in X$, there is an open ball s.t. $x_n \in B(r_z, z)$ for only finitely many indices. (if $\exists z$ st. $B(r, z)$ $\forall$ for any r contained infinitely many x_n then can construct a subsequence converging to z). Then $X = \bigcup_{z \in X} \{z\} \subseteq \bigcup_{z\in X} B(r_z, z) \subseteq X$ so $\{B(r_z, z)\}_{z \in X}$ is an open cover of X. Since X is compact there is a finite subcover $\{B(r_{z_i}, z_i)\}_{i=1}^{n}$. Then since $\{n \mid x_n \in B(r_{z_i}, z_i)\}$ is finite $\Rightarrow \{n \mid x_n \in \bigcup_{i=1}^{n} B(r_{z_i}, z_i)\}$ is finite but this is the same as (or greater than) $\{n \mid x_n \in X\}$ which is infinite. Contradiction. ■

Definition Let (X, d) be a metric space. A sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ is a Cauchy sequence if $\forall \varepsilon > 0$ there is an index N s.t. if $n, m > N$ then $d(x_n, x_m) < \varepsilon$.

Definition (X, d) is complete if every Cauchy sequence converges in X.

Theorem If (X, d) is compact then it is complete. (converse is not true: $\mathbb{R}$ complete not compact)

Lemma If a Cauchy sequence has a convergent subsequence, then the entire sequence converges.

Proof Let $\{x_{n_j}\}_{j=1}^{\infty}$ be a convergent subsequence. Let $z = \lim_{j\to\infty} x_{n_j}$. Want to show $\lim_{n\to\infty} x_n = z$.
Let $\varepsilon > 0$ be given. Choose N_1 so that $n_j > N_1 \Rightarrow d(z, x_{n_j}) < \frac{\varepsilon}{2}$. Now since $\{x_n\}$ Cauchy, pick N_2 s.t. $m, n > N_2 \Rightarrow d(x_m, x_n) < \frac{\varepsilon}{2}$. Claim: if $m > \max\{N_1, N_2\}$ then $d(x_m, z) < \varepsilon$.
Pick $n_j > \max\{N_1, N_2\}$. Then $d(x_m, z) \le d(x_m, x_{n_j}) + d(x_{n_j}, z) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. ■

Proof of theorem: BWP + Lemma. ■

Theorem $\mathbb{R}^n$ is complete wrt any norm metric.

Proof: Suppose $\{\vec{x}_m\}$ is a Cauchy sequence. Then $\{x_m\}$ is bounded: choose an index M s.t. $p, m \ge M \Rightarrow |\vec{x}_m - \vec{x}_p| < 1$. Then $\min\{\vec{x}_m\}_{m=1}^{M} - 1 \le \vec{x}_p \le \max\{\vec{x}_m\}_{m=1}^{M} + 1 \quad \forall p$.
By BWP, since $\{\vec{x}_m\}$ is contained in some closed interval, it has a convergent subsequence w/ a limit in the interval. By lemma, this means $\{\vec{x}_m\}$ converges so $\mathbb{R}^n$ is compact. ■

Definition Let $f: (X, d_1) \to (Y, d_2)$ be a function between metric spaces. We say that f is uniformly continuous if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\forall x, y \in X$, $d(x,y) < \delta \Rightarrow d_1(f(x), f(y)) < \varepsilon$.

Theorem If $f: (X, d_1) \to (Y, d_2)$ is continuous and X is compact, then f is uniformly cts.

Proof: By contradiction. Suppose f not unif. cts. Then $\exists \varepsilon > 0$ and $\forall \delta > 0$ s.t. $\exists x_\delta, z_\delta \in X$ s.t. $d_1(x_\delta, z_\delta) < \delta$ but $d_2(f(x_\delta), f(z_\delta)) \ge \varepsilon$. Take $\delta = \frac{1}{n}$. Then can find

$x_n, z_n \in X$ s.t. $d_1(x_n, z_n) < \frac{1}{n}$ but $d_2(f(x_n), f(z_n)) \geq \varepsilon$. By BWP, $\{x_n\}$ has a convergent subsequence $\{x_{n_j}\}$ let $\lim_{j\to\infty} x_{n_j} = a \in X$. Then

$d(z_{n_j}, a) \leq d(z_{n_j}, x_{n_j}) + d(x_{n_j}, a) < d(x_{n_j}, a) + \frac{1}{n} \to 0$

So $\lim_{j\to\infty} z_{n_j} = a \in X$. by continuity, $\lim_{j\to\infty} f(x_{n_j}) = f(a) = \lim_{j\to\infty} f(z_{n_j})$.

$\varepsilon \leq d_2(f(x_{n_j}), f(z_{n_j})) \leq d_2(f(x_{n_j}), f(a)) + d_2(f(a), f(z_{n_j})) \to 0$, a contradiction. ■