Lec 1/23

Monday, January 23, 2017 09:07

Definition (strongest) $\to$ Default. Let $S \subseteq \mathbb{R}^n$. $F: S \to \mathbb{R}$. We say that $\lim_{\vec{x}\to\vec{a}} F(\vec{x}) = L$ if
$\forall \varepsilon > 0,\ \exists \delta > 0$ s.t. $0 < |\vec{x} - \vec{a}| < \delta$ (any norm) $\Rightarrow$ $\vec{x} \in S$ and $|F(\vec{x}) - L| < \varepsilon$

Weaker Notion Definition Let $S \subseteq T \subseteq \mathbb{R}^n$, $F: T \to \mathbb{R}$. We say that $\lim_{\vec{x}\to\vec{a},\ \vec{x}\in S} F(\vec{x}) = L$ if
(implicitly assuming $\vec{a} \in \partial(S\setminus\{a\})$) $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $0 < |\vec{x} - \vec{a}| < \delta$ and $\vec{x} \in S \Rightarrow |F(\vec{x}) - L| < \varepsilon$

Still weaker notion Definition: Let $S \subseteq \mathbb{R}^2$, $F: S \to \mathbb{R}$. We say that $\lim_{y\to b}\lim_{x\to a} F(x,y) = L$ if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall y$ satisfying $0 < |y - b| < \delta$ there is a $\delta_y > 0$ s.t. $0 < |y-b| < \delta$ and $0 < |x - a| < \delta_y \Rightarrow (x,y) \in S$ and $|F(x,y) - L| < \varepsilon$.

Example: $F(x,y) = \dfrac{y+x}{y-x}$ $S = \{(x,y) : x \neq y\}$.

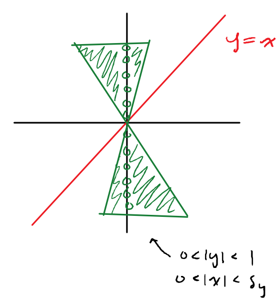
$$\lim_{y\to 0}\lim_{x\to 0} \frac{y+x}{y-x} = \lim_{y\to 0} \frac{y}{y} = 1. \qquad \lim_{x\to 0}\lim_{y\to 0} \frac{y+x}{y-x} = \lim_{x\to 0} \frac{x}{-x} = -1.$$

Rigorous proof of $\lim_{y\to 0}\lim_{x\to 0} \frac{y+x}{y-x} = 1$ using 3rd Def:
given $\varepsilon > 0$ take $\delta = 1$, $\delta_y = \frac{\varepsilon}{2+\varepsilon}|y|$. Then if $0 < |y| < 1$ and $0 < |x| < \frac{\varepsilon}{2+\varepsilon}|y| < |y| \Rightarrow (x,y) \in S$.

$$\frac{2|y|}{2+\varepsilon} = |y| - \frac{\varepsilon}{2+\varepsilon}|y| < |y| - |x| \le |y - x| \Rightarrow \frac{1}{|y-x|} < \frac{2+\varepsilon}{2|y|} < \frac{\varepsilon}{2|x|}$$

and $\left(|x| < \frac{\varepsilon}{2+\varepsilon}|y|\right)\frac{2+\varepsilon}{2|y||x|} = \dfrac{2+\varepsilon}{2|y|} < \dfrac{\varepsilon}{2|x|}$

now $\left|\dfrac{y+x}{y-x} - 1\right| = \dfrac{2|x|}{|y-x|} < \varepsilon$



Theorem: If $\lim_{(x,y)\to(a,b)} F(x,y) = L$ (by strong def) then $\lim_{y\to b}\lim_{x\to a} F(x,y) = \lim_{x\to a}\lim_{y\to b} F(x,y) = L$.

Proof: take $\delta = \delta_y$ s.t. $|(x,y) - (a,b)|_\infty < \delta \Rightarrow x \in S$, $|F(x,y) - L| < \varepsilon$.

Improved Theorem: If $S \subseteq \mathbb{R}^2$, $F: S \to \mathbb{R}$ and for some $r > 0$ $B(r,(a,b)) \cap \{(x,y) : x \neq a, y \neq b\} \subseteq S$
Then $\lim_{(x,y)\to(a,b),\,(x,y)\in S} F(x,y) = L$. Then $\lim_{y\to b}\lim_{x\to a} F(x,y) = \lim_{x\to a}\lim_{y\to b} F(x,y) = L$.

Proof: Same as above.

Application: With appropriate hypotheses on $f: S \subseteq \mathbb{R}^2 \to \mathbb{R}$ we have $\dfrac{\partial^2 f}{\partial y \partial x}(a,b) = \dfrac{\partial^2 f}{\partial x \partial y}(a,b)$

$$\frac{\partial^2 f}{\partial y \partial x}(a,b) = \lim_{y\to b} \frac{\frac{\partial f}{\partial x}(a,y) - \frac{\partial f}{\partial x}(a,b)}{y-b} = \lim_{y\to b} \frac{\lim \frac{f(x,y)-f(a,y)}{x-a} - \lim_{x\to a}\frac{f(x,b)-f(a,b)}{x-a}}{y-b}$$

$$= \lim\ \lim \frac{f(x,y) - f(a,y) - f(x,b) + f(a,b)}{(x-a)(y-b)}$$

$y \to b$

$$= \lim_{y \to b} \lim_{x \to a} \underbrace{\frac{f(x,y) - f(a,y) - f(x,b) + f(a,b)}{(x-a)(y-b)}}_{F(x,y)} \implies \frac{\partial^2 f}{\partial x \partial y} = \lim_{x \to a} \lim_{y \to b} F(x,y)$$

Theorem: (Heine-Borel) $A \subseteq \mathbb{R}^n$ is compact iff A is closed in $\mathbb{R}^n$ and bounded wrt any norm.

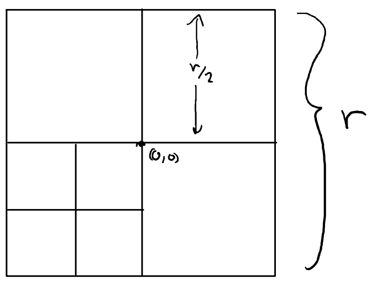
Proof: $\Rightarrow$ (already covered) ✓

$\Leftarrow$ reduces to showing that $\overline{B}_\infty(r, \vec{0})$ is compact. If A closed & bounded then A is closed subset of compact $\overline{B}_\infty(r, \vec{0})$ for large enough $r \Rightarrow A$ is compact.

Given an open cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of $\overline{B}_\infty(r, \vec{0})$ in $\mathbb{R}^n$, we want to find a finite subcover $\{U_{\alpha_i}\}_{i=1}^m$ s.t. $\overline{B}_\infty(r, \vec{0}) \subseteq \bigcup_{i=1}^m U_{\alpha_i}$.

Proof by contradiction. Suppose no finite subcollection satisfies that property.

$n=2$:



Subdivide $\overline{B}_\infty(r, \vec{0})$ into 2^n boxes of radius $r/2$.
at least one sub-box cannot be finitely covered.
Iterate this process (keep subdividing that box) to obtain a collection of boxes $\overline{B}^0 \supseteq \overline{B}^1 \supseteq \overline{B}^2 \supseteq \cdots$ so that

(1) $\overline{B}^j$ has radius $\frac{r}{2^j}$

(2) $\overline{B}^j$ cannot be finitely covered.

By nested boxes theorem, $\bigcap_{j=0}^{\infty} \overline{B}^j = \{\vec{c}\}$ which can be covered by a single open set, so one of the boxes must be able to be covered by one open set, a contradiction.