Lec 1/20

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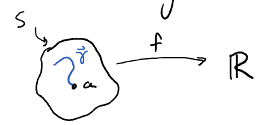
Theorem 5 in notes: Let $S \subseteq \mathbb{R}^n$, $f: S \to \mathbb{R}$. Then following statements are equiv:

(1) $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L$



(2) $\vec{a}$ is an interior point of $S \cup \{\vec{a}\}$ and

for any path $\vec{\gamma}: [0,1] \to S \cup \{\vec{a}\}$ s.t. $\vec{\gamma}^{-1}(\vec{a}) = 0$, we have $f \circ \vec{\gamma}: [0,1] \to \mathbb{R}$

and $\lim_{t \to 0^+} (f \circ \vec{\gamma})(t) = L$.

This is typically used to show $\lim_{\vec{x} \to \vec{a}} f(\vec{x})$ does not exist.

i.e. find two paths $\vec{\gamma}_1$ and $\vec{\gamma}_2 : [0,1] \longrightarrow S \cup \{\vec{a}\}$ satisfying (2)

and show that $\lim_{t \to 0^+} (f \circ \vec{\gamma}_1)(t) \neq \lim_{t \to 0^+} (f \circ \vec{\gamma}_2)(t)$.

Example:
$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } x \neq 0. \\ 0 & \text{if } x = 0. \end{cases}$$

show $\lim_{(x,y) \to (0,0)} f(x,y)$ DNE.

$\vec{\gamma}_1(t) = (0, t)$ $\qquad \lim_{t \to 0^+} (f \circ \vec{\gamma}_1)(t) = 0$

$\vec{\gamma}_2(t) = (t^2, t)$ $\qquad \lim_{t \to 0^+} (f \circ \vec{\gamma}_2)(t) = \lim_{t \to 0^+} \dfrac{t^3}{t^4 + t^2} = \lim_{t \to 0^+} \dfrac{6}{24t} = \infty$

Weak notion of limit

implicit in notation.

Definition 8 $T \subseteq \mathbb{R}^n$ $f: T \to \mathbb{R}$ $\boxed{S \subseteq T,\ \vec{a} \in \partial(S \setminus \{a\})}$. We define

$\lim_{\vec{x} \to \vec{a},\, \vec{x} \in S} f(\vec{x}) = L$ to mean that $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$0 < |\vec{x} - \vec{a}| < \delta$ and $\vec{x} \in S \implies |f(\vec{x}) - L| < \varepsilon$.

(comparable to 1-sided limits of 1-var calculus: $\lim_{x \to 0,\, x \in (0,\infty)} = \lim_{x \to 0^+}$, etc.)

Another notion of limit: iterated single-variable limits. (consider $n=2$)

$\lim_{y \to b} \lim_{x \to a} f(x,y) = L$ means fix y at some value $\neq b$, take limit of x, then take limit of y.

Precise Definition 10: $\lim_{y\to b}\lim_{x\to a} f(x,y) = L$ means that $\forall \varepsilon>0$, can find $\delta>0$ and for any y satisfying $0<|y-b|<\delta$, we can find a $\delta_y>0$ s.t. for any x satisfying $0<|x-a|<\delta_y$, $0<|y-b|<\delta$, we have $(x,y)\in \mathrm{dom}(f)$ and $|f(x,y)-L|<\varepsilon$.

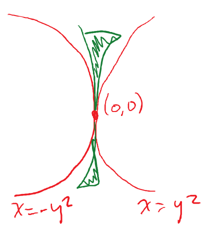
Theorem: if $\lim_{(x,y)\to(a,b)} f(x,y) = L$ then $\lim_{y\to b}\lim_{x\to a} f(x,y) = L$.

if $\lim_{y\to b}\lim_{x\to a} f(x,y)$ DNE then $\lim_{(x,y)\to(a,b)} f(x,y)$ DNE. (contrapositive)

Proof: Use the box norm in definition of limit. Given $\varepsilon>0$, find $\delta>0$ s.t. $0<|(x,y)-(a,b)|<\delta' \Rightarrow (x,y)\in \mathrm{dom}(f)$ and $|f(x,y)-L|<\varepsilon$.

then take $\delta = \delta_y = \delta'$

Example $f(x,y) = \dfrac{x^4}{y^4-x^2} = \dfrac{x^4}{(y^2-x)(y^2+x)}$



$\lim_{(x,y)\to(0,0)} f(x,y)$ DNE. but $\lim_{y\to 0}\lim_{x\to 0} f(x,y) = 0$.

$= \lim_{x\to 0}\lim_{y\to 0} f(x,y)$

#: region for iterated limit.

Given $\varepsilon>0$, take $\delta=1$, $\delta_y = \sqrt{\frac{\varepsilon}{1+\varepsilon}}\, y^2$

(it works, see notes)

Theorem: (Heine-Borel) Let $S\subseteq \mathbb{R}^n$, then the following are equivalent.

(1) S is a compact set

(2) S is closed in $\mathbb{R}^n$ and bounded with respect to any norm on $\mathbb{R}^n$.

Proof: (1)$\Rightarrow$(2): $\mathbb{R}^n$ Hausdorff $\Rightarrow$ S is closed in $\mathbb{R}^n$. also,

$S\subseteq \mathbb{R}^n = \bigcup_{n=1}^{\infty} B(n,\vec{0})$. Since S compact, there is a finite subcover $\{B(n_i,\vec{0})\}_{i=1}^{k}$. Then $S\subseteq B(\max\{n_i\}_{i=1}^{k}, \vec{0})$ so S is bounded.

(2)$\Rightarrow$(1): Recall some distinguishing properties of $\mathbb{R}$:

(a) Completion axiom: $\mathbb{R} = L\cup R$, $L\neq\emptyset\neq R$, $L<R \Rightarrow$ either L has a maximal point or R has a minimal point.

(b) any subset $S\subseteq\mathbb{R}$ which is bounded above has a least upper bound.

(c) similarly for g.l.b.

(d) nested intervals property & $\mathbb{R}$ has no infinitesimals. $\left(\lim_{n\to\infty} \frac{1}{n} = 0\right)$.

$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$, $I_j = [a_j, b_j]$, $\lim_{j\to\infty} b_j - a_j = 0$,

$\bigcap_{j=1}^{\infty} I_j = \{c\}$ singleton.

NIP $\Rightarrow$ Nested Boxes Property: let $B_1 \supseteq B_2 \supseteq \dots$ be boxes in $\mathbb{R}^n$,

$B_m = [a_{m_1}, b_{m_1}] \times \dots \times [a_{m_n}, b_{m_n}]$, and $\lim_{m\to\infty} (b_{m_j} - a_{m_j}) = 0 \quad \forall j \in \{1, \dots, n\}$

then $\bigcap_{m=1}^{\infty} B_m = \{\vec{c}\}$ singleton.

Proof Sketch: use box norm, show closed box is compact.

$S \subseteq$ compact $\Rightarrow$ closed sub sets of compact sets are compact.