Lec 1/19

Thursday, January 19, 2017 09:14

$S \subseteq \mathbb{R}^n \qquad f : S \to \mathbb{R}$

$\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L$

$n = 1$ "Definition:" (ambiguous) $\lim_{x \to a} f(x) = L$ if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\overbrace{0 < |x-a| < \delta \Rightarrow |f(x) - f(a)| < \delta}^{(*)}$

Good interpretation: $(*)$ means that $0 < |x-a| < \delta \Rightarrow x \in \mathrm{dom}(f)$ & $|f(x) - f(a)| < \varepsilon$

Bad interpretation: $(*)$ means that $0 < |x-a| < \delta$ & $x \in \mathrm{dom}(f) \Rightarrow |f(x) - f(a)| < \varepsilon$

↳ $\lim_{x \to -1} \sqrt{x} = \pi$. (vacuously true).

Less bad interp: Assume $a \in \partial(\mathrm{dom}(f) \setminus \{a\})$. Then $\forall \delta > 0$, $\{x \mid 0 < |x-a| < \delta\} \cap \mathrm{dom}(f) \neq \emptyset$

w/ this: $\lim_{x \to 0} \sqrt{x} = 0 \quad \lim_{x \to 0} \sqrt{-x} = 0 \quad \lim_{x \to 0} (\sqrt{x} + \sqrt{-x})$ not defined. (0 is only pt.)

"Counterexample" to L'Hôpital's rule: for $x \neq 0$, let $f(x) = \dfrac{x}{2 + x\sin(\frac{2}{x})}$ $\quad g(x) = f(x)\, e^{-\sin(\frac{1}{x})}$

Then $\lim_{x \to 0} f(x) = \lim_{x \to 0} g(x)$ but $\lim_{x \to 0} \dfrac{f(x)}{g(x)} = \lim_{x \to 0} e^{-\sin(\frac{1}{x})}$ does not exist.

$f'(x) = \alpha(x) \cos(\frac{1}{x}) \qquad g'(x) = \beta(x) \cos(\frac{1}{x})$ $\qquad$ for some $\alpha, \beta : (-\infty, 0) \cup (0, \infty) \to \mathbb{R}$.

$\lim_{x \to 0} \dfrac{\alpha(x)}{\beta(x)} = 0$.

suspect (true w/ Bad & less Bad interps) (false w/ good interp).

$\lim_{x \to 0} \dfrac{f'(x)}{g'(x)} = \lim_{x \to 0} \dfrac{\alpha(x)\cos(\frac{1}{x})}{\beta(x)\cos(\frac{1}{x})} = \lim_{x \to 0} \dfrac{\alpha(x)}{\beta(x)} = 0$

"Spivak" definition (bad). f is continuous at a if $\lim_{x \to a} f(x) = f(a)$.

equiv. to saying $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $|x - a| < \delta \Rightarrow x \in \mathrm{dom}(f)$ & $|f(x) - f(a)| < \varepsilon$.

Then $\sqrt{x}$ is not cts at 0.

Extreme Value thm: if $f(x)$ cts on $[a, b]$ (closed) then f attains a max & min on $[a,b]$.

(does not apply to $\sqrt{x}$ on $[0, 1]$).

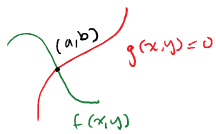
Theorem: $\lim_{x \to a} f(x) = f(a) \iff f$ defined on open interval $\ni a$ & f cts.

In $\mathbb{R}^n$ even more necessary to be careful about defs.

$n=2$. Suppose $f(a,b) = 0 = g(a,b)$. can we make sense of $\lim\limits_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)}$?

usually not.

Typically, $f(x,y)=0$, $g(x,y)=0$ define curves in the plane.



near the curve $g(x,y)=0$, the quotient blows up

on the curve $f(x,y)=0$, the quotient is 0.

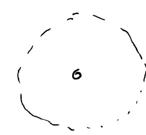
Good Definition: ($S \subseteq \mathbb{R}^n$, $h: S \to \mathbb{R}$). $\lim\limits_{\vec{x}\to\vec{a}} h(\vec{x}) = L$ if $\forall \varepsilon>0$, $\exists \delta>0$ s.t.

$0<|\vec{x}-\vec{a}|<\delta \Rightarrow x \in S$ and $|h(x)-L|<\varepsilon$.

Remark: This automatically excludes most limits of the type $\lim\limits_{\vec{x}\to\vec{a}} \frac{f(\vec{x})}{g(\vec{x})}$ where f, g are continuous functions with $f(\vec{a}) = 0 = g(\vec{a})$:

$$\lim_{\vec{x}\to\vec{a}} \underbrace{\frac{f(\vec{x})}{g(\vec{x})}}_{=h(\vec{x})} = L$$ requires that $\exists \delta>0$ s.t.

$0<|\vec{x}-\vec{a}|<\delta \Rightarrow \vec{x}\in \mathrm{dom}(h) \Rightarrow g(\vec{x}) \neq 0$.



Theorem: $\lim\limits_{\vec{x}\to\vec{a}} h(\vec{x}) = L$ is equivalent to the following: $\forall$ sequence $\{\vec{x}_n\}_{n=1}^{\infty}$ in $\mathrm{dom}(h)\setminus\{a\}$ and with $\lim\limits_{n\to\infty} \vec{x}_n = a$, we have $\lim\limits_{n\to\infty} h(\vec{x}_n) = L$. ($h$ is defined on some $B(\delta, a)\setminus\{a\}$).

Proof: $\Rightarrow$: Let $\varepsilon>0$ be given. Then find $\delta>0$ s.t. $0<|\vec{x}-\vec{a}|<\delta \Rightarrow \vec{x}\in\mathrm{dom}(h)$ & $|h(\vec{x})-L|<\varepsilon$

if $\lim\limits_{n\to\infty} \vec{x}_n = \vec{a}$, then for some N, $n>N \Rightarrow 0<|\vec{x}_n-\vec{a}|<\delta \Rightarrow \vec{x}\in\mathrm{dom}(h)$ & $|h(\vec{x})-L|<\varepsilon$

$\Leftarrow$: If not true then for some $\varepsilon>0$ and any $\frac{1}{n}<\delta$, can find $\vec{x}_n \in B(\frac{1}{n}, a)$ and $|h(\vec{x})-L|\geq\varepsilon$

so $\lim\limits_{n\to\infty} \vec{x}_n = a$ but $\lim\limits_{n\to\infty} h(\vec{x}_n) \neq L$.

Corollary: $\lim\limits_{\vec{x}\to\vec{a}} h(\vec{x}) = L \iff h$ is defined on some $B(\delta,a)\setminus\{a\}$ and for any continuous path $\vec{\gamma}:[0,1]\to\mathrm{dom}(h)$ with $\vec{\gamma}(0)=\vec{a}$, we have $\lim\limits_{t\to 0} h(\vec{\gamma}(t)) = L$.