Lec 1/18

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Definition: let (X,d) be a metric space (more generally: $(X,\mathcal{T})$ a topological space) an open cover of X is a collection of open sets $\{U_\alpha\}$ s.t. $X = \bigcup_\alpha U_\alpha$. A subcover is a collection $\{U_\beta\} \subseteq \{U_\alpha\}$ which is also an open cover of X.

Definition X is compact if any open cover of X has a finite subcover.

Example: Any finite metric/topological space is compact since the powerset of the space is finite so there are only finitely many open sets.

Nonexamples: $\mathbb{R}^n$: $\mathbb{R}^n = \bigcup_{m=1}^{\infty} B(m,\vec{0})$ (any standard metric) but if there were a finite subcover, $\mathbb{R}^n \subseteq B(\max(m),\vec{0})$.

$(0,1)$: $\{(\frac{1}{m},1)\}_{m=2}^{\infty}$ is an open cover w/ no finite subcover. (same situation as above).

Definition1 let $(X,\mathcal{T})$ be a topological space (not necessarily compact) and let $A \subseteq X$. we say A is a compact subset of X if $(A,\mathcal{T}_A)$ (subspace topology) is a compact topological space ($\mathcal{T}_A = \{U \cap A \mid U \in \mathcal{T}\}$).

Definition2 Let X be a topological/metric space, let $A \subseteq X$. we say that a collection of open sets $\{U_\alpha\}$ in X is an open cover of A in X if $A \subseteq \bigcup_\alpha U_\alpha$. we say $\{U_\beta\} \subseteq \{U_\alpha\}$ is a subcover if $A \subseteq \bigcup_\beta U_\beta$

Proposition: A is a compact subset of X iff any open cover of A in X has a finite subcover.

Proof: $A \subseteq \bigcup_\alpha U_\alpha \iff A = \left(\bigcup_\alpha U_\alpha\right) \cap A$

$A \subseteq \bigcup_{i=1}^{n} U_{\alpha_i} \iff A = \left(\bigcup_{i=1}^{n} U_{\alpha_i}\right) \cap A$

Remark: compactness is an absolute property (not relative to enclosing space, just topology)

if $A \subseteq X$, (top/met space)
$A \subseteq Y$, " "

& the subspace topologies on A coincide, then
A compact in $X \Leftrightarrow A$ compact in Y

Proposition: suppose $(X, \mathcal{T})$ is a compact top. space. Let $A \subseteq X$ be closed. Then A is compact.

Proof: suppose $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ is an open cover of A in X $(A \subseteq \bigcup_\alpha U_\alpha)$.

Then $X = A \cup (X \setminus A) \subseteq \left(\bigcup_\alpha U_\alpha\right) \cup (X \setminus A) \subseteq X$, This means $\{U_\alpha\}_{\alpha \in \mathcal{A}} \cup \{X \setminus A\}$ is an open cover of X. So there is a finite subcover $X = \bigcup_{i=1}^{n} U_{\alpha_i} \cup (X \setminus A)$.

$\Rightarrow A \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$. since $A \cap (X \setminus A) = \phi$.

is the converse true? i.e. if $A \subseteq X$ compact subset of X is A closed?

No! Example: Any finite top. space: $X = \{1, 2\}$ $\mathcal{T} = \{\phi, X, \{1\}\}$.

1 is compact but open.

Need additional property:

Definition we say that a topological space $(X, \mathcal{T})$ is Hausdorff if for any two distinct points $a, b \in X$, $a \neq b$, we can find two open sets $U, V \in \mathcal{T}$ s.t. $a \in U$, $b \in V$, $U \cap V = \phi$.

Proposition: Any metric space is Hausdorff.

Proof: Take balls w/ radius $\frac{d(b,a)}{2}$: $U = B\left(\frac{d(b,a)}{2}, a\right)$
$V = B\left(\frac{d(b,a)}{2}, b\right)$.

$U \cap V$ empty (o.w. if $z \in U \cap V$,
then $d(a,b) \leq d(a,z) + d(z,b) < \frac{d(b,a)}{2} + \frac{d(b,a)}{2} = d(b,a)$).

Theorem: let (X, τ) be a Hausdorff topological space. let $A \subseteq X$ be a compact subset. then A is closed in X.

Proof: A closed in $X \iff X \setminus A$ is open in X.

let $b \in X \setminus A$. we want to show b is an interior pt of $X \setminus A$ in X.

for each $a \in A$, pick open sets U_a and V_a s.t. $a \in U_a$, $b \in V_a$ $U_a \cap V_a = \emptyset$.

then $A = \bigcup_{a \in A} \{a\} \subseteq \bigcup_{a \in A} U_a$ i.e. $\{U_a\}_{a \in A}$ is an open cover of A in X.

so (by compactness) there is a finite subcover $\{U_{a_i}\}_{i=1}^{n}$ of $\{U_a\}_{a \in A}$ of A in X.

let $V = \bigcap_{i=1}^{n} V_{a_i}$. $b \in V$, and $V \cap A \subseteq V \cap \left(\bigcup_{i=1}^{n} U_{a_i}\right) = \bigcup_{i=1}^{n} (U_{a_i} \cap V)$

$$\subseteq \bigcup_{i=1}^{n} (U_{a_i} \cap V_{a_i}) = \emptyset.$$

this implies $V \subseteq X \setminus A$. $\Rightarrow$ b is an interior point, so $X \setminus A$ is open so A is closed.

Theorem: (Heine-Borel) A is a compact subset of $(\mathbb{R}^n, d)$ metric from a norm.

$\iff$ A is closed & bounded.

Note: This is not true if we take an arbitrary metric on $\mathbb{R}^n$ which is equiv. to the usual one.

Exercise (notes): (take-home midterm) if (X, d) any metric space then $\tilde{d}(x,y) = \frac{d(x,y)}{1 + d(x,y)}$ is an equivalent metric (not strongly). & $\tilde{d}(x,y) < 1$ $\forall x,y \in X$.

(we could say $\mathbb{R}^n$ is bounded, and it is closed, but it is not compact in itself).