Lec 1/17

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HW Help: $\frac{\sin(xy)}{x}$

Hold y fixed: $\lim_{x\to 0} \frac{\sin(xy)}{x} = \lim_{x\to 0} \frac{\sin(xy)}{xy} y$

$= \lim_{\theta\to 0} \frac{\sin\theta}{\theta} y = y.$

$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

Theorem $\vec{f} = (f_1, \dots, f_n) : (X,d) \to \mathbb{R}^n$ is continuous iff $f_1, \dots, f_n : (X,d) \to \mathbb{R}$ are continuous

Proof: $\Rightarrow$: f_i is the composite $(X,d) \xrightarrow{\vec{f}} \mathbb{R}^n \xrightarrow{\pi_i} \mathbb{R}$

so f_i is cts if $\vec{f}$ is cts.

$\Leftarrow$: suppose f_i is cts $\forall i \in \{1, \dots, n\}$.

to show that $\vec{f}$ is cts, it suffices to show that $\vec{f}^{-1}$(open ball) is open in (X,d).

Use box metric on $\mathbb{R}^n$. $B_\infty(r, \vec{a}) = (a_1 - r, a_1 + r) \times \cdots \times (a_n - r, a_n + r)$

$\vec{f}^{-1}(B_\infty(r, \vec{a})) = f_1^{-1}((a_1 - r, a_1 + r)) \cap \cdots \cap f_n^{-1}((a_n - r, a_n + r))$

(open) $\cap \cdots \cap$ (open)   (finite intersection)

So $\vec{f}^{-1}(B_\infty(r, \vec{a}))$ is open

Problem 1, §1.2.

(b). $S = \{(x,y) \mid x^2 - x \le y \le 0\}$ is a closed subset of $\mathbb{R}^2$.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ $f(x,y) = x^2 - x - y$

$\pi_2 : \mathbb{R}^2 \to \mathbb{R}$ $\pi_2(x,y) = y$.

$S = \{(x,y) \mid f(x,y) \le 0\} \cap \{(x,y) \mid \pi_2(x,y) \le 0\}$

$= f^{-1}((-\infty, 0]) \cap \pi_2^{-1}((-\infty, 0])$

$=$ closed.

Definition: suppose (X,d) is a metric space, $a \in X$, $\{x_n\}_{n=1}^{\infty}$ sequence in X.

Then we say that $\lim_{n\to\infty} x_n = a$ if $\lim_{n\to\infty} d(x_n, a) = 0$.

for any $\varepsilon > 0$, $x_n \in B(\varepsilon, a)$ for all $n \geq N$.

Limits $\overset{\text{of seqs}}{\vee}$ are unique in metric spaces but not necessarily in topological spaces.

Proof by contradiction: Suppose that $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} x_n = b \neq a$.

$$\underbrace{d(a,b)}_{\text{fixed positive}} \leq \underbrace{d(a, x_n) + d(x_n, b)}_{\to\, 0 \text{ in } \lim_{n\to\infty}}$$

contradiction.

Proposition let $\{x_m\}_{m=1}^{\infty}$ be a sequence in a metric space (X,d).

then $\lim_{m\to\infty} x_m = a$ iff $\forall$ neighborhood N of a, we can find an index M s.t. $x_m \in N$ when $m \geq M$.

Corollary limits of sequences are the same for equivalent metrics.

Corollary^2 $\lim_{m\to\infty} \vec{x}_m = \vec{a}$ in $\mathbb{R}^n$ iff $\lim_{m\to\infty} x_{m_i} = a_i$ in $\mathbb{R}$ $\forall\, i \in \{1, \dots, n\}$.

Proof: use box metric in $\mathbb{R}^n$

Proposition: Let (X,d) be a metric space and $A \subseteq X$. Then A is closed in X iff for any convergent sequence $\{a_m\} \in A$ $\lim_{m\to\infty} a_m \in A$.

Proof: $\Rightarrow$: Suppose A is closed and let $\lim_{m\to\infty} a_m \notin A$ so it's $\in X \setminus A$.

Then $\lim_{m\to\infty} a_m$ is a boundary point of A in X which is not in A.

(Since any open ball $\overset{\text{about the limit}}{\vee}$ contains $a_m \in A$ and $\lim_{m\to\infty} a_m \notin A$.) Contradiction.

$\Leftarrow$: Suppose that $\{a_m\}$ convergent sequence in $A \Rightarrow \lim_{m\to\infty} a_m \in A$.

Need to show A is closed in X. $\Leftrightarrow$ $\partial A \subseteq A$

Suppose $b \in \partial A$. then $\forall$ positive integers m, $\overset{\text{define } a_m \text{ so}}{\vee}$ there is an $a_m \in B(\frac{1}{m}, b)$

then $\lim_{m\to\infty} a_m = b$ so $b \in A$ so $\partial A \subseteq A$.

Proposition: $f: (X, d_1) \to (X, d_2)$ is continuous at $a \in A$ iff $\forall$ sequence $\{a_m\} \in X$ s.t. $\lim_{m\to\infty} a_m = a$, we have $\lim_{m\to\infty} f(a_m) = f(a)$.

Proof: If f is continuous at a, then for any neighborhood N of $f(a)$, $f^{-1}(N)$ is a neighborhood of a. Then if $\lim_{m\to\infty} a_m = a$, for some index M, $a_m \in f^{-1}(N)$ for $m \geq M$. $\Rightarrow$ $f(a_m) \in N$ for $m \geq M$. so $\lim_{m\to\infty} f(a_m) = f(a)$.

Conversely: suppose f is not cts at a. In that case, $\exists\, \varepsilon > 0$ s.t. for any $\delta > 0$ s.t. $B_1(\delta, a) \not\subseteq f^{-1}(B_2(\varepsilon, f(a)))$.

Choose $\delta = \frac{1}{m}$, Can find $x_m \in B_1(\frac{1}{m}, a) \not\subseteq f^{-1}(B_2(\varepsilon, f(a)))$

then $\lim_{m\to\infty} x_m = a$ but $d_2(f(x_m), f(a))$.