Lec 1/13

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Theorem: $f:(X,d_1)\to(Y,d_2)$ then the following are equiv.

(1) f cont on X

(2) $f^{-1}(\text{open}) = \text{open}$

(3) $f^{-1}(\text{closed}) = \text{closed}$

proposition If (X,d) metric space and $S\subseteq X$, then (S,d) is also a metric space.

(restrict d to $S\times S$). has the subspace topology:

open in (S,d) = (open in (X,d)) $\cap S$

closed " = closed " "

Proposition: If $f(X,d_1)\to(Y,d_2)$ is cts, $S\subseteq X$, $T\subseteq Y$ and $f(S)\subseteq T$, then f restricts to a continuous function

$f|_{S\to T}:(S,d_1)\to(T,d_2)$

Proposition: (1) Composites of cts funcs are cts.

(2) identity, constant functions are cts.

Proof: $f:(X,d_1)\to(Y,d_2)$, $g:(Y,d_2)\to(Z,d_3)$ cts.

then if $W\subseteq Z$ is open in (Z,d_3) then $g^{-1}(W)\subseteq Y$ is open in (Y,d_2)

so $f^{-1}(g^{-1}(W))$ is open in (X,d_1) $\left((g\circ f)^{-1}(\text{open}) = \text{open}\right)$.

Lemma 0: $f:(X,d_1)\to(Y,d_2)$ is continuous iff f^{-1}(any open ball in (Y,d_2)) is open.

Lemma 1 any open interval in $\mathbb{R}$ is an open set in $\mathbb{R}$ (under usual metric).

Proof: (a,b) is an o ball $B\left(\frac{b-a}{2}, \frac{b+a}{2}\right)$

(a,∞) is $\bigcup_{n=1}^{\infty}(a,a+n)$

$(-\infty,a)$ is $\bigcup_{n=1}^{\infty}(a-n,a)$

Proposition: $f:(X,d_1)\to\mathbb{R}$ is continuous iff for any $a\in\mathbb{R}$, $f^{-1}(-\infty,a)$ and $f^{-1}(a,\infty)$ are open in (X,d).

Proof: by lemma 0 f cont $\Leftrightarrow$ $f^{-1}((a,b))$ open in (X,d) $\forall a<b\in\mathbb{R}$.

$\Leftarrow$:
$$f^{-1}((a,b)) = f^{-1}((a,\infty)\cap(-\infty,b)) = f^{-1}((a,\infty))\cap f^{-1}((-\infty,b))$$

$\Rightarrow$: consequence of lemma 1.

Proposition: $inv:\mathbb{R}\setminus\{0\}\to\mathbb{R}$ $inv(x)=\frac{1}{x}$ is continuous

Proof: we only need to check that $inv^{-1}((-\infty,a))$, $inv^{-1}((a,\infty))$ are open (case by case)

1) $a>0$: $inv^{-1}((a,\infty)) = (0,\frac{1}{a})$ ✓

$inv^{-1}((-\infty,a)) = inv^{-1}((-\infty,0)\cup(0,a)) = (-\infty,0)\cup(\frac{1}{a},\infty)$ ✓

Next, show that (1) $add:\mathbb{R}^2\to\mathbb{R}$ $add(x,y)=x+y$ are cts.
(2) $mul:\mathbb{R}^2\to\mathbb{R}$ $mul(x,y)=xy$

Metric approach to (1).

Definition $f:(X,d_1)\to(Y,d_2)$ is distance preserving if $d_2(f(u),f(v)) = d_1(u,v)$.
is (nonstrictly) distance reducing if $d_2(f(u),f(v))\le d_1(u,v)$

distance preserving $\Rightarrow$ nonstrict distance reducing.

Proposition A distance reducing map is continuous.

Proof: $d_1(x,a)<\delta \Rightarrow d_2(f(x),f(a))<\varepsilon$

take $\delta=\varepsilon$

Theorem: add is continuous.

Proof: $|x+y|-|a+b| \le |x-a|+|y-b|$

$|a+b|-|x+y| \le |a-x|+|b-y|$

$$\Rightarrow \underbrace{\big||x+y|-|a+b|\big|}_{} \le \underbrace{|x-a|+|y-b|}_{}$$

$$\Rightarrow \underbrace{\big||x+y| - |a+b|\big|}_{\text{normal dist btwn } (x+y) \text{ and } (a+b).} \le \underbrace{|x-a| + |y-b|}_{\text{taxicab dist btwn } (x,y) \text{ and } (a,b).}$$

add : $(\mathbb{R}^2, ||_1) \to (\mathbb{R}, ||)$ is distance reducing, so it's continuous

Lemma 2: Let σ be a permutation of $\{1, \dots, n\}$. Let $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ be the corresponding permutation of coordinates, so
$\sigma(x_1, \dots, x_n) = \sigma(x_{\sigma(1)}, \dots, x_{\sigma(n)})$. σ is cts.

Proof: (1) metric proof: σ is distance preserving wrt d_p.
since $|\sigma(\vec{x})|_p = |\vec{x}|_p$ for $p \in [1, \infty]$.

(2) topological proof: $\sigma^{-1}(B_p(r, \vec{a})) = B_p(r, \sigma^{-1}(\vec{a}))$.
Hence inverse images of open sets are open.

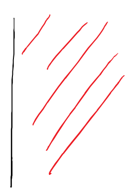
Lemma 3: Let $\pi_i : \mathbb{R}^n \to \mathbb{R}$ be the ith projection $\pi_i(\vec{x}) = x_i$.
Then π_i is continuous.

Proof: (1) metric proof: π_i is distance reducing wrt any $||_p$. $\Rightarrow$ cts.

(2) $\pi_{\sigma(1)} = \pi_i \circ \sigma$. Since we can find permutations σ s.t. $\sigma(1) = i$, it suffices to show that π_1 is continuous.
So it suffices to show that $\pi_1^{-1}((a, \infty))$ and $\pi_1^{-1}((-\infty, a))$ are open in $(\mathbb{R}^n, ||_\infty)$

$$\pi_1^{-1}((a, \infty)) = (a, \infty) \times (-\infty, \infty)^{n-1}$$
$$= \bigcup_{m=1}^{\infty} (a, a+2m) \times (-m, m)^{n-1} = B_\infty(n, (a+n, 0, \dots, 0)) \text{ is open. } \checkmark$$

Corollary: open Horizontal & vertical Half planes are open sets in $\mathbb{R}^2$



$x = a$

$$= H_a^+ = \{(x,y) \mid x > a\} = \pi_1^{-1}((a, \infty))$$

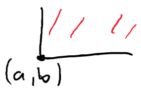
$$H_a^- = \{(x,y) \mid x < a\} = \pi_1^{-1}((-\infty, a))$$

$y = b$

$$= V_b^+ = \{(x,y) \mid y > b\} = \pi_2^{-1}((b, \infty))$$

$V_b^- = \{(x,y) \mid y < b\}$ $\quad \pi_2^{-1}((-\infty, b))$

Corollary²: Open horiz/vert quadrants are open sets in $\mathbb{R}^2$



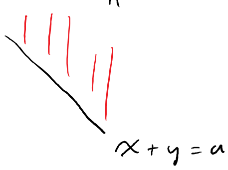
$= Q^{++}_{(a,b)} = \{(x,y) \mid x>a, y>b\} = H_a^+ \cap V_b^+$

$= Q^{+-}_{(a,b)} = H_a^+ \cap V_b^-$

Topological proof of open sets that add is cts:

NTS: $add^{-1}((a,\infty))$, $add^{-1}((-\infty,a))$ are open.

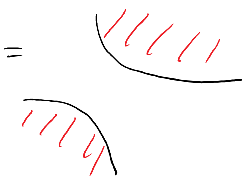
$\|$



$= \bigcup_{x+y=a} Q^{++}_{(x,y)}$

$= \bigcup_{x+y=a} Q^{--}_{(x,y)}$

for mul: $mul^{-1}((a,\infty)) =$



$= \left(\bigcup_{\substack{xy=a \\ x>0, y>0}} Q^{++}_{(x,y)} \right) \cup \left(\bigcup_{\substack{xy=a \\ x<0, y<0}} Q^{--}_{(x,y)} \right)$