Lec 1/11

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(X, d) metric space, $S \subseteq X$

(1) $a \in S$ $\underbrace{\text{interior pt of } S \text{ in } (X,d)}$ if $\exists r$ s.t. $B(r,a) \subseteq S$.

$S^{int} = \{a \in S : a \text{ is } \quad \}$

(2) $a \in X$ $\underbrace{\text{is a boundary pt of } S \text{ in } (X,d)}$ if $\forall r$ $B(r,a) \cap S \neq \emptyset$, $B(r,a) \cap X \setminus S \neq \emptyset$

$\partial S = \{a \in X : a \quad \} = \partial X \setminus S$

$X = S^{int} \sqcup \partial S \sqcup (X \setminus S)^{int}$

(3) S is open in (X,d) if $S = S^{int}$

S is closed in (X,d) if $\partial S \subseteq S$

$\overline{S}$ (closure of S) $= S \cup \partial S$

Proposition (basic properties of open sets in a metric space (X,d)):

(1) X & $\emptyset$ are open in (X,d).

(2) $\{U_\alpha\}$ arbitrary collection of open sets in (X,d), $\bigcup_\alpha U_\alpha$ is open in (X,d).

(3) if U, V open in (X,d) then $U \cap V$ is open in (X,d).

Proof: (1) ✓

(2) if $a \in \bigcup_\alpha U_\alpha$ then $a \in U_{\alpha_0}$ for some α_0. So $\exists r>0$ s.t. $B(r,a) \subseteq U_{\alpha_0} \subseteq \bigcup_\alpha U_\alpha$

so a is an interior pt. ✓

(3) let $a \in U, V$. Choose $r_1, r_2 > 0$ s.t. $B(r_1, a) \subseteq U$, $B(r_2, a) \subseteq V$. then

$B(\min(r_1, r_2), a) \subseteq U \cap V$. a is interior pt. ✓

(3) $\Rightarrow$ (3'): any finite intersection of open sets in (X,d) is open.

this all in $(X,d) = \mathbb{R}$

but infinite intersections are not open in general. Consider $\{(-\frac{1}{n}, \frac{1}{n})\}_{n=1}^{\infty}$. $\bigcap_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n}) = \{0\}$.

Proposition: S is open in (X,d) iff $X \setminus S$ is closed in (X,d)

Proof: $X = S^{int} \sqcup \partial S \sqcup (X \setminus S)^{int}$ ∎

Examples δ: disc. metric on X, every subset of X is both open & closed.

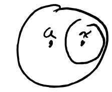
Proposition: S is open in (X,d) iff $S = \bigcup_\alpha B(r_\alpha, a_\alpha)$ arbitrary union of open balls.

Proof: If S is open, then $S = S^{int}$, so $\forall a \in S$ pick a radius $r > 0$ s.t. $B(r_a, a) \subseteq S$.

Then $S = \{a \in S\} \subseteq \bigcup_{a \in S} B(r_a, a) \subseteq S$. ✓

Lemma: $B(r,a)$ is open.

Proof: Suppose $x \in B(r,a)$. Let $s = r - d(x,a) > 0$



Then $B(s,x) \subseteq B(r,a)$ by triangle inequality:

if $y \in B(s,x)$ then $d(y,a) \leq d(y,x) + d(x,a) < s + d(x,a) = r$ ✓

If S is an arbitrary union of open balls, it is open by (2). ✓

Proposition: (basic properties of closed sets in a metric space (X,d)):

(1) X and ϕ are closed in (X,d).

(2) $\{F_\alpha\}$ arbitrary collection of closed sets, then $\bigcap_\alpha F_\alpha$ is closed in (X,d)

(3) If F, H are closed in (X,d) then $F \cup H$ is closed in (X,d).

Proof: (de Morgan's laws) + proposition 0

↳ $X \setminus (F \cup H) = X \setminus F \cap X \setminus H$ and $X \setminus (F \cap H) = X \setminus F \cup X \setminus H$

Definition A topological space $(X, \mathcal{T})$ is a set with a specified collection $\mathcal{T} \subseteq \mathcal{P}(X)$ of subsets of X satisfying:

(1) $X, \phi \in \mathcal{T}$

(2) if $U_\alpha \in \mathcal{T}$ for $\alpha \in A$ then $\bigcup_\alpha U_\alpha \in \mathcal{T}$ (→ large index set)

(3) $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$

$\mathcal{T}$ is called the topology of X.

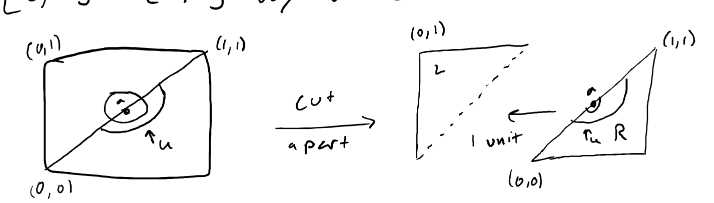
Any metric d on X specifies a topology $\mathcal{T}_d$ on X.

two metrics d_1, d_2 on X are equivalent if $\mathcal{T}_{d_1} = \mathcal{T}_{d_2}$ on X. ($\mathcal{T}_d$ is the open sets specified by metric d)

Remark: you can specify a topology on X by specifying a collection of sets declared to be closed (satisfying (1),(2),(3)) and $\mathcal{T}$ is built using complements.

Example:

$X = [0,1] \times [0,1]$ w/ $d = d_2$.



now with a new metric d'

$$d'(\vec{x}, \vec{y}) = \begin{cases} d_2(\vec{x}, \vec{y}) & \text{if } \vec{x}, \vec{y} \text{ both in } L \text{ or both in } R. \\ \sqrt{(|x_1 - y_1| + 1)^2 + (x_2 - y_2)^2} & \text{otherwise} \end{cases}$$

$$d'(\vec{x},\vec{y}) = \begin{cases} d_2(\bar{x},y) & \text{if } x,y \text{ both in L or both in R.} \\ \sqrt{(|x_1-y_1|+1)^2+(x_2-y_2)^2} & \text{otherwise.} \end{cases}$$

u is open in (X,d') but not in (X,d)