Lec 1/10

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Proof of Triangle inequality for ℓ_2

$|\vec{x}|_{(2)} = \sqrt{\sum_{i=1}^{n} x_i^2}$ $\quad \vec{x} \in \mathbb{R}^n$

wts: $|\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}|$

this is equiv. to cauchy-schwarz inequality:

$\vec{x} \cdot \vec{y} \leq |\vec{x}||\vec{y}|$ $\quad \cos\theta = \frac{\vec{x} \cdot \vec{y}}{|\vec{x}||\vec{y}|}$ where θ is 'angle' between $\vec{x}$ and $\vec{y}$

Note: $\vec{x} \cdot \vec{y} = \sum_{i=1}^{n} x_i y_i$

$(|\vec{x} + \vec{y}|)^2 \overset{?}{\leq} (|\vec{x}| + |\vec{y}|)^2$

$(\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) \overset{?}{\leq} |\vec{x}|^2 + |\vec{y}|^2 + 2|\vec{x}||\vec{y}|$

$\underset{|\vec{x}|^2}{\underset{\parallel}{\vec{x} \cdot \vec{x}}} + \underset{|\vec{y}|^2}{\underset{\parallel}{\vec{y} \cdot \vec{y}}} + 2\vec{x} \cdot \vec{y}$

so $2\vec{x} \cdot \vec{y} \overset{?}{\leq} 2|\vec{x}||\vec{y}|$

yes by CSI. so go backwards.

Proof of T.I. (& CSI) from scratch

Strategy: (1) prove CSI in $\mathbb{R}^2$

(2) $\Rightarrow$ T.I. in $\mathbb{R}^2$ (equiv)

(3) $\Rightarrow$ T.I. in $\mathbb{R}^n$

(4) $\Rightarrow$ CSI in $\mathbb{R}^n$ (equiv)

(1) $\vec{x} = (x_1, x_2)$ $\quad \vec{y} = (y_1, y_2)$

$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2$ so wts $x_1 y_1 + x_2 y_2 \leq \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}$

which is implied by:

$(x_1y_1 + x_2y_2)^2 \le (x_1^2 + x_2^2)(y_1^2 + y_2^2)$ (since RhS is positive)

$\Updownarrow$ bunch of algebra

$0 \le x_1^2 y_1^2 + x_2^2 y_2^2 - 2x_1x_2y_1y_2 = (x_1y_1 - x_2y_2)^2$ so it works out.

(3) Prove Δ in $\mathbb{R}^n$: induction on n.

$n = 1$ ✓

$n = 2$ ✓

let $\vec{x} = (x_1, x_2, \dots, x_{n-1}, x_n)$ $\quad \vec{y} = (y_1, y_2, \dots, y_{n-1}, y_n)$

$\vec{x}' = (x_1, x_2, \dots, x_{n-1})$ $\quad \vec{y}' = (y_1, y_2, \dots, y_{n-1})$

now $|\vec{x} + \vec{y}| = \sqrt{|\vec{x}' + \vec{y}'|^2 + (x_n + y_n)^2} \underset{\text{ind. hyp.}}{\le} \sqrt{(|\vec{x}'| + |\vec{y}'|)^2 + (x_n + y_n)^2}$

$= \left| \left(|\vec{x}'| + |\vec{y}'|, \; x_n + y_n \right) \right| \overset{\Delta \text{ in } \mathbb{R}^2}{\le} \left| (|\vec{x}'|, x_n) \right| + \left| (|\vec{y}'|, y_n) \right|$

$= \sqrt{|\vec{x}'|^2 + x_n^2} + \sqrt{|\vec{y}'|^2 + y_n} = |\vec{x}| + |\vec{y}|$ ∎

Norms in "$\mathbb{R}^\infty$" in which vectors are sequences of real nums.

$\vec{x} = \{x_i\}_{i=1}^{\infty}$

$|\vec{x}|_p = \sqrt[p]{\sum_{i=1}^{\infty} |x_i|^p}$ where $\sum_{i=1}^{\infty} |x_i|^p$ converges. (depends on p)

Defn: $\ell^p = \left\{ \{x_i\}_{i=1}^{\infty} \;\middle|\; \sum_{i=1}^{\infty} |x_i|^p \text{ converges} \right\}$. Now we can define the norm & metric on ℓ^p to be as above.

Triangle inequality follows from taking limits of finite dimension cases.

ℓ^2 is known as "Hilbert Space" – Fourier series.

a 2π periodic func. $f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$

corresponds to a point $(a_0, a_1, b_1, a_2, b_2, \dots)$ in ℓ^2.

Now what about $p = \infty$?

$\vec{x} \in \ell^\infty \Rightarrow |\vec{x}|_\infty = \sup\{|x_i|\}_{i=1}^\infty$ so ℓ^∞ = bounded sequences.

Discrete Metric X any set, $\delta : X \times X \to [0, \infty)$

$$\delta(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

(1) & (2) hold, what about Δ ineq?

$\delta(x,z) \le \delta(x,y) + \delta(y,z)$?

LHS	RHS
0	0
1	1
	2

only possible failure, but if RHS is 0 then $x = z = y$ so $\delta(x,z) = 0$.
So it's impossible.

Lemma: if (X, δ) is discrete metric space & (Y, d) is any metric space, $f : X \to Y$ is continuous everywhere.

Proof: let $a \in X$, $\varepsilon > 0$. wtf $\lambda > 0$ s.t. $\delta(a,x) < \lambda \Rightarrow d(f(x), f(a)) < \varepsilon$.

Pick $\lambda = 1$. So $\delta(a,x) < 1 \Rightarrow x = a \Rightarrow d(f(x), f(a)) = 0 < \varepsilon$.

Open Balls

Let (X,d) be a metric space. Then Defn $B(r,a) = \{x \in X \mid d(a,r) < r\}$ open ball. $r > 0$.

in (X, δ), $B(r,a) = \begin{cases} X & \text{if } r > 1 \\ \{a\} & \text{if } r \le 1 \end{cases}$

Definitions: let $S \subseteq (X, d)$. we say

(1) a is an interior point if $\exists r > 0$ s.t. $B(r,a) \subseteq S$.

(2) S is open in X if $\forall a \in S$, a is an interior point. $S^{int} = \{a \in S \mid a \text{ is an interior pt}\}$

(3) a is a boundary point if $\forall r > 0$ $B(r,a)$ contains both points in S and in $X \setminus S$.

$\partial S = \{a \in X \mid a \text{ is a boundary point of } S\}$.

$\partial S = \partial(X \setminus S)$

..

$X = S^{int} \sqcup \partial S \sqcup (X \setminus S)^{int}$ where $\sqcup$ is disjoint union.

$S \setminus \partial S = S^{int}$ and $S \setminus S^{int} \subseteq \partial S$

(4) S is closed in X if $\partial S \subseteq S$.

$\overline{S} = S \cup \partial S$ is closed.

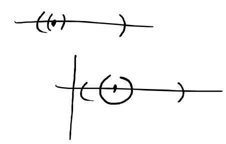
(5) S is a neighborhood of a if $a \in S^{int}$

Openness / Closedness depends on underlying metric space.

all notions above are relative notions.

ex: $(0,1)$ is open in $\mathbb{R}$.

but is not open in $\mathbb{C} = \mathbb{R}^2$



$[0,1)$ is open in $[0,\infty)$.