Lec 11/14

Monday, November 14, 2016 9:09 AM

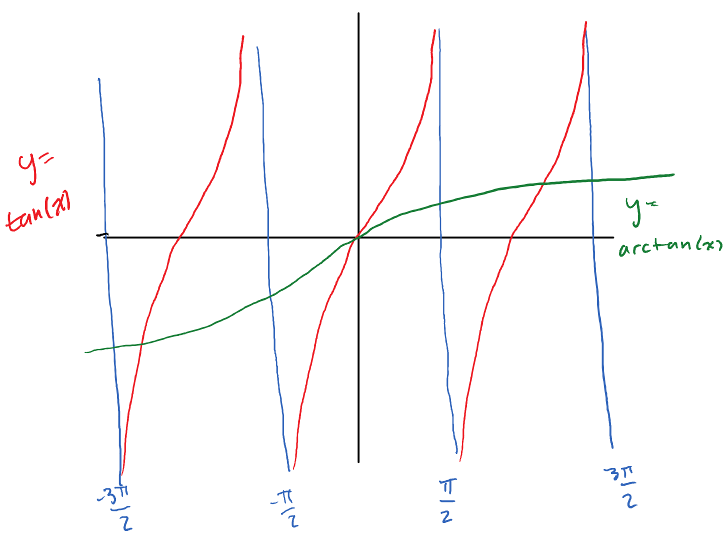
$\tan = \frac{\sin}{\cos}$ $\sec = \frac{1}{\cos}$ $\cot = \frac{1}{\tan}$ $\csc = \frac{1}{\sin}$

$\cos^2(x) + \sin^2(x) = 1 \Rightarrow 1 + \tan^2(x) = \sec^2(x)$

$\frac{d}{dx}(\tan(x)) = \frac{\cos(x)\cos(x) + \sin(x)\sin(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$

$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$

$\tan(x+\pi) = \tan(x)$



$\cos(x) = 0$ for $x = (2K+1)\frac{\pi}{2}$

$\tan: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ 1-1 & onto.

has inverse

$\tan^{-1} = \arctan: \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$

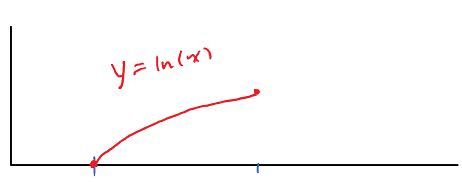
$\lim_{x\to\pm\infty} \arctan(x) = \pm\frac{\pi}{2}$

Note: $\tan^{-1}(x) \neq \frac{1}{\tan(x)}$, etc.

$\frac{d}{dx}(\arctan(x)) = \frac{1}{\tan'(\arctan(x))} = \frac{1}{\sec^2(\arctan(x))} = \frac{1}{1+\tan^2(\arctan(x))} = \frac{1}{1+x^2}$

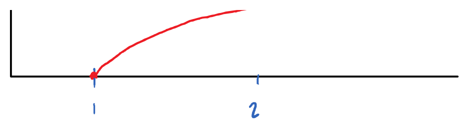
So $\int \frac{1}{1+x^2}\,dx = \arctan(x)$.

Problem: find the length of the graph $y = \ln(x)$ $1 \le x \le 2$



general formula for arc length of $y = f(x)$

$\int_a^b \sqrt{1 + [f'(x)]^2}\,dx$



in this case, $f'(x) = \frac{1}{x}$, so the length is

$$L = \int_1^2 \sqrt{1 + \frac{1}{x^2}}\, dx = \int_1^2 \sqrt{\frac{x^2+1}{x^2}}\, dx = \int_1^2 \frac{\sqrt{x^2+1}}{x}\, dx$$

general technique: $\sqrt{\text{quadratic}} \Rightarrow x = A\sin(t),\ B\tan(t),\ \text{or}\ C\sec(t)$

$\sqrt{A^2 - x^2} \longrightarrow A\cos(t)$

$\sqrt{B^2 + x^2} \longrightarrow B\sec(t)$

$\sqrt{x^2 - C^2} \longrightarrow C\tan(t)$

If there is a linear term, must complete the square.

$u = x + b$

$x = \tan(t) \Rightarrow$

$dx = \sec^2(t)\,dt$

$$L = \int_{\arctan(1) = \frac{\pi}{4}}^{\arctan(2)} \frac{\sqrt{\tan^2(t)+1}}{\tan(t)} \sec^2(t)\,dt = \int_{\frac{\pi}{4}}^{\arctan(2)} \frac{\sec^3(t)}{\tan(t)}\,dt = \int_{\frac{\pi}{4}}^{\arctan(2)} \frac{1}{\cos^2(t)\sin(t)}\,dt$$

general technique: $\int [\sin(x)]^m [\cos(x)]^n\,dx$

if m odd, $u = \cos(x)$

if n odd, $u = \sin(x)$

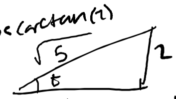
and use pythagorean identity.

$u = \cos(t)$

$\frac{du}{-\sin(t)} = dt$

$\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$\cos(\arctan(2))$



$\sin(t) = \frac{1}{\sqrt{5}}$

$$L = \int_{\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{5}}} \frac{1}{u^2\sin(t)} \frac{du}{-\sin(t)} = \int_{\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{5}}} \frac{du}{-\sin^2(t)u^2} = \int_{\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{5}}} \frac{du}{-(1-\cos^2(t))u^2} = \int_{\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{5}}} \frac{du}{u^2(u^2-1)}$$

~~general technique~~: partial fractions decomposition

1) factor denominator $u^2(u-1)(u+1)$

2) general form: $\frac{A}{u+1} + \frac{B}{u-1} + \frac{C}{u} + \frac{D}{u^2} = \frac{1}{u^2(u-1)(u+1)}$

multiply by $u^2(u-1)(u+1)$

$$A u^2(u-1) + B u^2(u+1) + C u(u^2-1) + D(u^2-1) = 1$$

Two ways to compute A, B, C, D:

(1) substitute numerical values of u.

(2) use linear algebra (compare coeff. of like powers).

try (1): $u=0 \Rightarrow -D = 1$ so $D = -1$.

$u=1 \Rightarrow 2B = 1$ so $B = \frac{1}{2}$

$u=-1 \Rightarrow -2A = 1$ so $A = -\frac{1}{2}$

try (2): $A u^3 + Bu^3 + Cu^3 = 0$

$-\frac{1}{2}u^3 + \frac{1}{2}u^3 + Cu^3 = 0 \Rightarrow C = 0.$

So $$L = -\frac{1}{2}\int_{\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{5}}} \frac{1}{u+1}\,du + \frac{1}{2}\int_{\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{5}}} \frac{1}{u-1}\,du + \int_{\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{5}}} \frac{-1}{u^2}\,du$$

$$L = -\frac{1}{2}\log|u+1| + \frac{1}{2}\log|u-1| + \frac{1}{u}\Big]_{\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{5}}}$$