Lec 8/24

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$$T(n) = \begin{cases} 4n^2 + 2n & \text{if } n \text{ even} \\ 3n & \text{if } n \text{ odd} \end{cases}$$

$T(n) \in \Theta(n^2 \mid n \text{ even})$

this means:
$\exists c, n_0$ s.t. $\forall n \geq n_0$ where $P(n)$, $T(n) \leq c f(n)$

Conditional Asymptotics.

let $P(n)$ be a predicate. We write $T(n) \in O(f(n) \mid P(n))$

$$O(f(n) \mid P(n)) \not\subseteq O(f(n)) \quad \text{in general}$$

What P work and allow $O(f(n) \mid P(n)) \subseteq O(f(n))$?

Smooth functions:

f is smooth iff f is asymptotically nondecreasing and $f(2n) \in O(f(n))$

$\exists N$ s.t. $f(m) \geq f(n)$ whenever $m \geq n \geq N$.

a smooth function grows slowly.

Since f asymp. nondec, $f(2n) \geq f(n)$, but $f(2n) \in O(f(n))$

ex: $\log n$, $n \log n$, n^2, n^3 smooth.

n^2 asymp nondec. $(2n)^2 = 4n^2 \in O(n^2)$ ✓.

but 2^n not smooth: $2^{2n} = 4^n \notin O(2^n)$. ×

any polynomial is smooth.

Smooth $\Rightarrow$ $f(bn) \in O(f(n))$ $\forall b \in \mathbb{N}$. Prove by induction.

<u>Proof</u>: for $b = 1, 2$ obvious

assuming for $b-1$, for any $b>2$, $f(bn) \overset{\text{f nondecreasing}}{\in} O\big(f(2(b-1)n)\big) \overset{\text{f smooth}}{\subseteq} O(f((b-1)n)) \overset{\text{ind. Hyp.}}{\subseteq} O(f(n))$.

<u>Theorem 5</u>: If $T(n) \in O(f(n) \mid n \text{ a power of } b)$, $b \geq 2$, $T(n)$ is asymp. nondecreasing, $f(n)$ is smooth, then $T(n) \in O(f(n))$.

<u>Proof</u>: $\exists k$ s.t. $b^k \leq n < b^{k+1}$.

Then $T(n) \overset{\text{nondec}}{\leq} T(b^{k+1}) \overset{\text{Hyp.}}{\leq} c_1 f(b^{k+1}) \overset{\text{f smooth}}{\leq} c_2 c_1 f(b^k) \overset{\text{nondec}}{\leq} c_2 c_1 f(n)$.

for sufficiently large n.

So $T(n) \in O(f(n))$. ∎

This theorem holds for Ω and Θ too.

<u>Application</u>:

$T(n) = 2T(\lfloor \frac{n}{2} \rfloor) + n$

If n is a power of 2,

$T(n) = 2T(\frac{n}{2}) + n$

$$T(n) \in \Theta(n \log n \mid n \text{ a power of } 2)$$

$$\Rightarrow \quad T(n) \in \Theta(n \log n)$$

$$\frac{n^2}{4} \le \overbrace{1 + 2 + \cdots + \underbrace{\frac{n}{2} + \cdots + n}_{n/2 \text{ terms} \ge \frac{n}{2}}}^{n \text{ terms} \le n} \le n^2 \qquad \Rightarrow \qquad \sum_{i=1}^{n} i \in \Theta(n^2).$$

for constant k, $\quad \sum_{i=1}^{n} i^k \in \Theta(n^{k+1})$ since

$$\frac{n^{k+1}}{2^{k+1}} \le \overbrace{1 + 2^k + \underbrace{\cdots + n^k}_{\frac{n}{2} \text{ terms} \ge \frac{n^k}{2^k}}}^{n \text{ terms} \le n^k} \le n\, n^k$$

$$\sum_{i=0}^{n} a^i = \frac{a^{n+1} - 1}{a - 1}$$

when $a > 1$, $\quad \sum_{i=0}^{n} a^i \in \Theta(a^n)$

when $a < 1$, $\quad \sum_{i=0}^{n} a^i \in \Theta(1)$

$$\int_m^n f \le \sum_{i=m}^{n} f(i) \le \int_m^n f + \max(f(m), f(n))$$

as long as f is monotone.

and if $\quad \int_m^n f \in \Omega(\max(f(m), f(n)))$

then $\sum_{i=m}^{n} f(i) \in \Theta\left(\int_m^n f\right)$

ex: $\sum_{i=m}^{n} \frac{1}{i} \in \Theta\left(\int_m^n \frac{1}{x}\, \partial x\right) = \Theta\left(\log n - \log m\right)$