Lec 4/4

Wednesday, April 4, 2018 12:39

Arithmetical Heierarchy

DFNs: a predicate is a mapping from ω^k to $\{T, F\}$

A predicate is recursive if it's turing-computable.

Let $\Sigma_0 = \{\{x : R(x)\} : R \text{ is a recursive predicate}\}$

= the set of recursive languages.

$\Sigma_1 = \{\{x : \exists y \text{ s.t. } R(x, y)\} : R \text{ is a recursive predicate}\}$.

= the set of r.e. languages (y is the step count).

$\Sigma_2 = \{\{x : \exists y_1 \text{ s.t. } \forall y_2\, R(x, y_1, y_2)\} : R \text{ recursive}\}$

$\Sigma_3 = \{\{x : \exists y_1 \text{ s.t. } \forall y_2, \exists y_3 \text{ s.t. } R(x, y_1, y_2, y_3)\} : R \text{ recursive}\}$.

⋮

$\exists$ if n odd, $\forall$ if n even.

$\Sigma_n = \{\{x : \exists y_1 \forall y_2 \exists y_3 \cdots Q_n y_n \; R(x, y_1, \ldots, y_n)\} : R \text{ recursive}\}$.

Define $\forall n \geq 0$, $\Pi_n = \{\bar{A} : A \in \Sigma_n\}$

Thus $\Pi_0 = \Sigma_0$

but $\bar{K} \in \Pi_1 \setminus \Sigma_1$, $K \in \Sigma_1 \setminus \Pi_1$

(in general, $\Sigma_n \neq \Pi_n \;\; \forall n \geq 1$).

Thus, e.g.

$$\Pi_2 = \{ \{x : \neg \exists y_1 \forall y_2\ R(x, y_1, y_2)\} : R \text{ recursive} \}$$

$$= \{ \{x : \forall y_1 \exists y_2\ \neg R(x, y_1, y_2) : R \text{ recursive} \}$$

$$= \{ \{x : \forall y_1 \exists y_2\ R(x, y_1, y_2) : R \text{ recursive} \}$$

In general, $$\Pi_n = \{ \{x : \forall y_1 \exists y_2 \cdots \underset{\substack{\forall \text{ if } n \text{ odd} \\ \exists \text{ if } n \text{ even}}}{Q_n} y_n\ R(x, y_1, \ldots, y_n) \} : R \text{ recursive} \}$$

Also define $\Delta_n = \Sigma_n \cap \Pi_n$

Proposition: $\forall n \geq 0$, $\Sigma_n \subsetneq \Delta_{n+1}$ and $\Pi_n \subsetneq \Delta_{n+1}$

pf obvious (add quantified variables to ignore).

Dfn: L is arithmetical if $\exists n$ s.t. $L \in \Sigma_n$.

is there a non arithmetical language?

Yes, only countably many languages are countable.

Also, $L = \{\langle n, x \rangle : x \in \emptyset^{(n)}\}$ is not arithmetical.

Claim: $\forall n\ [\emptyset^{(n)} <_T L]$.

Pf: $\emptyset^{(n)} \leq_T L$ since $x \in \emptyset^{(n)} \iff \langle n, x \rangle \in L$.

and $L \not\leq_T \emptyset^{(n)}$ since $\emptyset^{(n)} < \emptyset^{(n+1)} \leq_T L$, so if $L \leq \emptyset^{(n)}$ then $L \not\leq_T L$.

Recall: $\text{EMPTY} = \{e : W_e = \emptyset\} \in \Pi_1$

$$\text{FIN} = \{e : (\exists n)(\forall m > n)[m \notin W_e]\}$$

$$= \{ e : \exists n\ \forall m\ \forall s\ \ M_e^s(m)\uparrow \}$$

$$= \{ e : \exists n\ \forall \langle m, s \rangle\ (m \leq n \text{ or } M_e^s(m)\uparrow) \} \in \Sigma_2$$