Recall A and B are incomparable if $A \not\leq_T B$ and $B \not\leq_T A$.

Kleene-Post $\exists$ incomparable languages A and B s.t. $A \leq_T K$ and $B \leq_T K$.
(this implies $A <_T K$ and $B <_T K$).

Proof We'll construct incomparable A and B in stages I.e. We'll construct a sequence of finite-length binary strings:

$$\alpha_0 \subset \alpha_1 \subset \alpha_2 \subset \cdots$$

where $\alpha_j \subset \alpha_{j+1}$ means α_j is a proper prefix of α_{j+1}.

Then A will be defined by its characteristic function

$$\chi_A = \lim_{j \to \infty} \alpha_j \ . \quad \chi_A(n) = \alpha_{n+1}(n).$$

Analogously, we'll have $\beta_0 \subset \beta_1 \subset \cdots$ and let $\chi_B = \lim_{j\to\infty} \beta_j$.

We'll build A & B to satisfy these requirements:

if this is true $\forall e$, then $A \not\leq_T B$. $\longrightarrow$ R_{2e+1} : $\chi_A \neq \Phi_e^B$ (i.e. $A \not\leq_T B$ via the e^{th} oracle TM).

similarly for this one. $\longrightarrow$ R_{2e+2} : $\chi_B \neq \Phi_e^A$

Stage 0: Define $\alpha_0 = \beta_0 = \Lambda$ (the empty string).

Stage 2e+1: Our goal is to satisfy R_{2e+1}.

The notation $\Phi_{e,t}^{\sigma}(n)\downarrow$ means that the e^{th} oracle TM with oracle C (where σ is a finite prefix of C) accepts n in t steps and never scans cell $|\sigma|+1$.

We write $\Phi_{e,t}^{\sigma}(n)\downarrow = 1$ if the TM outputs 1.

Let $n = |\alpha_{2e}| + 1$. Now if $(\exists \sigma)(\exists t)[\beta_{2e} \subset \sigma$ and $\Phi_{e,t}^{\sigma}(n)\downarrow = 1]$

Then we'll define $\alpha_{2e+1} = \alpha_{2e} {}^\wedge 0$ so $n \in A$ (i.e. α_{2e} concatenated w/ 0).

and $\beta_{2e+1} = \sigma$. Thus n is a witness to R_{2e+1}.

otherwise we'll define $\alpha_{2e+1} = \alpha_{2e} {}^\wedge 1$ and $\beta_{2e+1} = \sigma$.

Stage 2e+2 Analogous, but reversing the roles of A and B.