Recursion Theorem:

Fixed Point form: Let σ be a total recursive function. Then $\exists i$ s.t. $\varphi_i = \varphi_{\sigma(i)}$.

Self-referential form: Let f be a partial recursive function. Then

$\exists i$ s.t. $\forall j$, $\varphi_i(j) = f(\langle i, j \rangle)$.

Proof of self-referential form: By S_{mn} theorem, $\exists$ a total computable σ s.t. $\forall i, j$, $\varphi_{\sigma(i)}(j) = f(\langle i, j \rangle)$. Then by the fixed-point form, $\exists\, i$ s.t. $\varphi_{i_0} = \varphi_{\sigma(i_0)}$, so $\varphi_{i_0}(j) = f(\langle i_0, j \rangle)$. $\square$

Exercise: assuming SR form, prove FP form.

Some applications of self-referential form:

① let $f(\langle i, j \rangle) = i$. Then $\exists i$ s.t. $\varphi_i(j) = i$ ("knows its own index").

② Say that T.M. M_i is minimal if $\varphi_i = \varphi_j \Rightarrow i < j$.
(or $\varphi_i \neq \varphi_j \;\; \forall j < i$).

Let $MIN = \{ i : M_i \text{ is minimal} \}$.

MIN is Not R.E. Assume it were. Then $\exists$ an enumerator for MIN.

Let $f(\langle i, j \rangle) = \varphi_m(j)$ where m is the first number greater than i which is enumerated. Then f is partial recursive.

So by self-referential form, $\exists i_0$ s.t. $\forall j$, $\varphi_{i_0}(j) = f(\langle i_0, j \rangle) = \varphi_{m_0}(j)$.

but $i_0 < m_0$ so $m_0 \notin MIN$, a contradiction $\square$

(1) Is there a 1-1 correspondence between K and EMPTY? Yes, they're both countable. What about a computable 1-1 correspondence?

No, since $K \in$ r.e. and EMPTY $\notin$ r.e., and having a 1-1 correspondence would mean $K \leq_m$ EMPTY and EMPTY $\leq_m K$ by f and g, where

$f(e)$ is the 1-1 correspondence if $e \in K$ and it's some non-empty otherwise

$g(e)$ is the inverse of the 1-1 if $e \in$ EMPTY and it's some non-K o.w.

actually these are not computable.

using 1-1 correspondence & an enumerator for K to get an enumerator for EMPTY, contradiction

So there is no 1-1 correspondence.

(2) Define $\leq_m$.

(3) Let $D = \{\langle i, j, k \rangle : i \in W_j \setminus W_k\}$.

is D recursive? is D r.e.?

NO: $i \mapsto \langle i, i, \underset{\in}{\overset{\text{EMPTY}}{x}} \rangle$ reduces K to D.

No: $i \mapsto \langle i, \underset{\in}{\overset{\text{TOT}}{x}}, i \rangle$ reduces $\overline{K}$ to D.

(4) Let $E = 2\omega$, P = set of primes.

i) Does $E \leq_m P$? ii) Does $P \leq_m E$?

Yes there is a computable 1-1 correspondence since both are recursive.

for 3, could also use $f(i) = \langle i, \overset{\mathrm{TOT}}{\downarrow}{a}, b \rangle$ where $W_b = K$.