1 Review

Last time we introduced the following objects:

Name	Notation	Example
Markov Process	X_t	Ornstein-Uhlenbeck process
Stationary Measure	μ	$N(0,1)$
Markov Semigroup	$(P_t f)(x) = \mathbb{E}_x[f(X_t)]$	$(P_t f)(x) = \mathbb{E}[f(e^{-t}x + \sqrt{1-e^{-2t}}Z)]$
Generator	$Lf = \lim_{t \searrow 0} \frac{P_t f - f}{t}$	$(Lf)(x) = f''(x) - xf'(x)$
Dirichlet Energy	$\mathcal{E}(f,g) = \langle f, -Lg \rangle_\mu$	$\mathcal{E}(f,g) = \mathbb{E}[\nabla f(Z) \cdot \nabla g(Z)]$

We also saw a few properties of these things. Perhaps the most important was the "covariance lemma" which states that

$$\mathrm{Cov}_\mu[f,g] = \int_0^\infty \mathcal{E}(f, P_t g)\, dt.$$

This can be proved using the heat equation $\partial_t P_t = LP_t$. We also said that X_t satisfies a "Poincaré inequality" with constant C if

$$\mathrm{Var}_\mu[f] \le C\mathcal{E}(f,f)$$

for all $f \in L^2(\mu)$. This is reminiscent of the Poincare inequality from PDEs. We saw that the Ornstein-Uhlenbeck process satisfies a Poincaré inequality with constant 1. In other words,

$$\mathrm{Var}[f(Z)] \le \mathbb{E}[|\nabla f(Z)|^2].$$

Let's see a quick application of this to Gaussian polymers.

2 Application of Poincaré Inequality

Recall that in the Gaussian polymer model, the minimal energy is given by

$$E_n = \inf_p \left\{ -\sum_{v \in p} g_v \right\}$$

where g_v are i.i.d. $N(0,1)$ at each vertex $v \in \mathbb{Z}^2$, and p ranges over all length-n polymers (i.e. graphs of $\mathbb{Z}$-random walks starting at 0 and taking n steps). Notice that

$$\frac{\partial E_n}{\partial g_v} = -\mathbf{1}_{\{v \in \text{ optimal path}\}}.$$

Thus

$$\begin{aligned}
|\nabla E_n|^2 &= \sum_v \left(\frac{\partial E_n}{\partial g_v} \right)^2 \\
&= \sum_v \mathbf{1}_{\{v \in \text{ optimal path}\}} \\
&= \text{number of vertices in optimal path} \\
&= n+1.
\end{aligned}$$

So the Gaussian Poincaré inequality gives $\mathrm{Var}(E_n) \le n+1$.

3 Spectral Interpretation

L is a negative semidefinite (self-adjoint) operator on $L^2(\mu)$, so $-L$ is positive semidefinite. We will assume that we can order the eigenvalues as

$$0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots,$$

with a corresponding sequence of orthonormal eigenfunctions

$$u_0, u_1, u_2, \ldots$$

This can be done in the case of the Ornstein-Uhlenbeck process, as well as all other things we will consider. Notice that $u_0 \equiv 1$ since the generator kills constant functions. By the spectral theorem, the eigenfunctions form an orthonormal basis for $L^2(\mu)$, so we can write

$$f = \sum_{k=0}^{\infty} \langle u_k, f \rangle_\mu u_k,$$

which implies that

$$\mathbb{E}_\mu[f^2] = \sum_{k=0}^{\infty} \langle u_k, f \rangle_\mu^2.$$

Since $u_0 \equiv 1$, we also have $\mathbb{E}_\mu[f] = \langle u_0, f \rangle_\mu$. Thus we can write

$$\mathrm{Var}_\mu[f] = \sum_{k=1}^{\infty} \langle u_k, f \rangle_\mu^2.$$

Also notice that the Dirichlet energy becomes

$$\mathcal{E}(f, f) = \langle f, -Lf \rangle_\mu = \sum_{k=0}^{\infty} \lambda_k \langle u_k, f \rangle_\mu^2 = \sum_{k=1}^{\infty} \lambda_k \langle u_k, f \rangle_\mu^2.$$

Therefore, since $\lambda_k \geq \lambda_1$ for $k \geq 1$, the optimal constant in the Poincaré inequality is $\frac{1}{\lambda_1}$. In other words, we always know that

$$\mathrm{Var}_\mu[f] \leq \frac{1}{\lambda_1} \mathcal{E}(f, f).$$

4 Superconcentration

We say that a function f is ϵ-superconcentrated if

$$\mathrm{Var}_\mu[f] \leq \frac{\epsilon}{\lambda_1} \mathcal{E}(f, f).$$

You should think of ϵ as being small in the sense that $\epsilon \to 0$ as $n \to \infty$ (there is always some n lurking in the background). In this case, we can just say that f is superconcentrated.

For example, the maximum of n i.i.d. Gaussians is superconcentrated. One can prove that

$$\mathrm{Var}\left[\max_{1 \leq i \leq n} Z_i\right] \sim \frac{C}{\log n},$$

whereas we have

$$\frac{\partial}{\partial Z_i} \max_{1 \leq i \leq n} Z_i = \mathbf{1}_{\{Z_i \text{ is the maximum}\}},$$

and since there will be a unique maximum almost surely, the Dirichlet energy of the maximum is 1. This shows that the maximum is $\frac{C}{\log n}$-superconcentrated.

On the other hand, any linear function like $\frac{Z_1 + \cdots + Z_n}{\sqrt{n}}$ is not superconcentrated The $\sqrt{n}$ in the denominator is immaterial for this fact, since it will scale out of both sides in the Poincaré inequality.

5 Chaos

Assume that $\lambda_1 > 0$. Then, recalling the heat equation $\partial_t P_t = LP_t$ which implies that $P_t = e^{tL}$, we have

$$\begin{aligned}
\mathcal{E}(f, P_t f) &= \langle f, -Le^{-tL} f\rangle_\mu \\
&= \sum_{k=1}^{\infty} \lambda_k e^{-\lambda_k t} \langle u_k, f\rangle_\mu^2 \\
&\leq e^{-\lambda_1 t} \sum_{k=1}^{\infty} \langle u_k, f\rangle_\mu^2 \\
&= e^{-\lambda_1 t} \mathcal{E}(f, f).
\end{aligned}$$

If the Dirichlet energy is a way to measure the similarity between two functions, this says that the similarity decreases exponentially as we perturb the noise. We say that f is (ϵ, δ)-chaotic if for all $t \geq \delta$,

$$\mathcal{E}(f, P_t f) \leq \epsilon e^{-\lambda_1 t} \mathcal{E}(f, f).$$

Again, think of $\epsilon, \delta \to 0$ as $n \to \infty$. This says that the similarity decreases even more quickly than expected.

Let's see how this looks for Gaussian polymers. Recall the notation $g^t = e^{-t} g + \sqrt{1 - e^{-2t}} g'$, where g' is an independent Gaussian. Let $\hat{p}$ be the optimal path in the environment (g_v), and let $\hat{p}^t$ be the optimal path in the environment (g_v^t). Then we have

$$\begin{aligned}
\mathcal{E}(E_n, P_t E_n) &= \mathbb{E}[\nabla E_n \cdot \nabla P_t E_n] && \text{(deinition of } \mathcal{E} \text{ for OU)} \\
&= e^{-t} \mathbb{E}[\nabla E_n \cdot P_t \nabla E_n] && \text{(since } \nabla \circ P_t = e^{-t} P_t \circ \nabla) \\
&= e^{-t} \mathbb{E}[\nabla E_n(g) \cdot \nabla E_n(g^t)] && \text{(equivalent form of } P_t) \\
&= e^{-t} \sum_{v \in V} \mathbb{P}[v \in \hat{p} \text{ and } v \in \hat{p}^t] && \text{(derivative of } E_n \text{ from page 1)} \\
&= e^{-t} \mathbb{E}[|\hat{p} \cap \hat{p}^t|].
\end{aligned}$$

Since $\mathcal{E}(E_n, E_n) = n + 1$ from page 1, to say that E_n is (ϵ, δ)-chaotic is the same as saying that for all $t \geq \delta$,

$$\mathbb{E}[|\hat{p} \cap \hat{p}^t|] \leq \epsilon(n + 1).$$

6 Equivalence Between Superconcentration and Chaos

Theorem 1.

(a) *If f is ϵ-superconcentrated, then f is $\left(\frac{\epsilon e^{\lambda_1 \delta}}{\lambda_1 \delta}, \delta\right)$-chaotic for any $\delta > 0$.*

(b) *If f is (ϵ, δ)-chaotic, then f is $(\epsilon + \lambda_1 \delta)$-superconcentrated.*

Thus, if λ_1 is not large (as a function of n), then superconcentration and chaos are equivalent, since the smallness of the parameters are equivlent. To see this for part (a), choose $\delta = \sqrt{\epsilon}$.

The idea of the proof is actually quite simple; we just need to study the covariance lemma

$$\operatorname{Var}_\mu[f] = \int_0^\infty \mathcal{E}(f, P_t f)\, dt.$$

If chaos holds, the $\mathcal{E}(f, P_t f)$ decreases rapidly to zero, so $\operatorname{Var}_\mu[f]$ must be small. This shows that chaos implies superconcentration.

On the other hand, one can show that $\mathcal{E}(f, P_t f)$ is a non-negative decreasing function of t. So if the whole integral is small, $\mathcal{E}(f, P_t f)$ must decrease rapidly. This shows that superconcentration implies chaos.

Using these ideas, you can probabl prove the theorem yourself, and the formulas above are what come out of the algebra you have to do.

7 Proof of Equivalence Theorem

First, recall that

$$\mathcal{E}(f, P_t f) = \sum_{k=1}^{\infty} \lambda_k e^{-\lambda_k t} \langle u_k f, \rangle_\mu^2.$$

This implies that $\mathcal{E}(f, P_t f)$ is nonnegative for all t and is a decreasing function of t. In fact, it also implies that $e^{\lambda_1 t}\mathcal{E}(f, P_t f)$ is nonincreasing.

Now suppose that f is ϵ-superconcentrated. Then

$$\frac{\epsilon}{\lambda_1}\mathcal{E}(f, f) \geq \mathrm{Var}_\mu[f] = \int_0^\infty \mathcal{E}(f, P_t f)\, dt.$$

By the nonnegativity and decreasingness of $\mathcal{E}(f, P_t f)$, we obtain for any $\delta > 0$ that

$$\frac{\epsilon}{\lambda_1}\mathcal{E}(f, f) \geq \int_0^\delta \mathcal{E}(f, P_t f)\, dt \geq \delta \mathcal{E}(f, P_\delta f).$$

Doing some algebra, and multiplying both sides by $e^{\lambda_1 \delta}$ gives

$$e^{\lambda_1 \delta}\mathcal{E}(f, P_\delta f) \leq \frac{\epsilon e^{\lambda_1 \delta}}{\lambda_1 \delta}\mathcal{E}(f, f).$$

Now using the nonincreasingness of $e^{\lambda_1 t}\mathcal{E}(f, P_t f)$ gives

$$e^{\lambda_1 t}\mathcal{E}(f, P_t f) \leq \frac{\epsilon e^{\lambda_1 \delta}}{\lambda_1 \delta}\mathcal{E}(f, f).$$

This proves that f is $\left(\frac{\epsilon e^{\lambda_1 \delta}}{\lambda_1 \delta}, \delta\right)$-chaotic.

Now suppose that f is (ϵ, δ)-chaotic. By the decreasingness of $\mathcal{E}(f, P_t f)$, we have

$$\begin{aligned}
\mathrm{Var}_\mu[f] &= \int_0^\infty \mathcal{E}(f, P_t f)\, dt \\
&\leq \int_0^\delta \mathcal{E}(f, f)\, dt + \int_\delta^\infty \epsilon e^{-\lambda_1 t}\mathcal{E}(f, f)\, dt \\
&\leq \delta \mathcal{E}(f, f) + \frac{\epsilon}{\lambda_1}\mathcal{E}(f, f) \\
&= \frac{\epsilon + \lambda_1 \delta}{\lambda_1}\mathcal{E}(f, f).
\end{aligned}$$

This shows that f is $(\epsilon + \lambda_1 \delta)$-superconcentrated.

8 Application

For the Gaussian polymer, we calculated that $\mathcal{E}(E_n, P_t E_n) = e^{-t}\mathbb{E}[|\hat{p} \cap \hat{p}^t|]$, which includes the earlier result that $\mathcal{E}(E_n, E_n) = n + 1$. Thus, in this case, superconcentration of E_n means tht

$$\mathrm{Var}[E_n] = o(n),$$

and chaos of E_n means that

$$\mathbb{E}\left[\left|\hat{p}_n \cap \hat{p}_n^{t_n}\right|\right] = o(n)$$

for some $t_n \to 0$. The theorem thus says that the two facts above are equivalent. Notice that we still haven't proved either one of these facts! Some time (and effort) will be dedicated to proving one or the other, but at least we don't need to prove both of them from scratch.