These notes are for a talk in the UC Berkeley student probability seminar, and are based mainly on [5], as well as some background material from various sources [1, 2, 3, 4]. Nothing in these notes constitutes original work by me.

1 Introduction

We will study probability measures of the form

$$d\nu^{\mathcal{E}}(\phi) \propto \exp\left(-\lambda \int_{\Omega} V(\phi_x)\, dx\right) d\nu_m^{\mathrm{GFF}}(\phi), \tag{1}$$

where $\lambda > 0$, $\Omega = \mathbb{T}^2$ is a 2d torus, and ν_m^{GFF} is the massive continuum Gaussian free field on Ω with mass $m \geq 1$. We will consider two cases, either $V(\phi_x) = \exp(\sqrt{\beta}\phi_x)$ or $V(\phi_x) = \cosh(\sqrt{\beta}\phi_x)$, in which cases we call this the *Liouville* or *sinh-Gordon* model respectively, and set $\mathcal{E} = \mathrm{Lv}$ or $\mathcal{E} = \mathrm{ShG}$.

Briefly, the base measure ν_m^{GFF} captures the behavior of a planar network of *coupled harmonic oscillators*. It can be thought of as the ground state of the quantum system of atoms in a planar lattice, each of which will be offset from their expected position by some amount, in such a way that the neighboring atoms stay close to one another. The effect of the mass $m \geq 1$ is to force each atom to be relatively close to its expected position, although in the present case this is a mild effect (to be discussed later). Note that the equation (1) does not actually make sense as it stands, since the GFF is a random *distribution*, not a function, but let us ignore this for now.

Let us first consider the Liouville model ν^{Lv}. The integral in the exponent,

$$\int_{\Omega} \exp\left(\sqrt{\beta}\phi_x\right) dx =: \mathcal{Z}^{\beta}, \tag{2}$$

is the partition function of the *Gaussian Multiplicative Chaos*, which is a Gibbs measure on Ω which uses the *value* of the GFF sample as the energy function. This means that it assigns higher weight to areas of Ω where the atoms are higher above their expected positions. By an energy vs entropy calculation, it can be shown that the GMC is concentrated on a particular *atypical level set* (also called *thick points*) of the GFF. The partition function $\mathcal{Z}^{\beta}$ should be thought of as a "soft-max", or a proxy for the maximum or other high values of the GFF. By the way, the paper [5] only considers the so-called L^2 phase, i.e. $\beta \in (0, 4\pi)$. This is the phase where the GMC can be defined as a limit in L^2 of a particular martingale measure defined in terms of the discretized/regularized GFF. It should be noted that the paper [6] considers the full L^1 phase, i.e. $\beta \in (0, 8\pi)$ but only studies ν^{Lv}, not ν^{ShG}; this seems to be primarily because exp is increasing but cosh is not.

Since the model (1) is a tilt by $e^{-\lambda \mathcal{Z}^{\beta}}$ of the GFF, where $\lambda > 0$, the Liouville model should be understood as a version of the GFF where the large-deviation fluctuations of the atoms are suppressed. This is because large values of $\mathcal{Z}_{\beta}$ mean that the GFF has a large proportion of thick points, and so $e^{-\lambda \mathcal{Z}^{\beta}}$ is larger when there are fewer thick points. When considering the Sinh-Gordon model ν^{ShG}, the interpretation is that it should be like a GFF but with fewer thick and $-$thick points. Since thick points are rare anyways, maybe the following result is not so surprising:

Theorem 1 (Informal)**.** *$\nu^{\mathcal{E}}$ and ν_m^{GFF} can be coupled such that their difference is somewhat smooth.*

2 Properly defining things

Again, (1) doesn't make sense because the GFF is not a function. Instead, we will work with a discretized version of (1), defined on the lattice $\Omega_{\varepsilon} = \mathbb{T}^2 \cap \varepsilon\mathbb{Z}^2$, with nearest-neighbor interactions. We will then need to define the discretized version of each part of (1), in particular we will need to understand the discretized version of the base measure ν_m^{GFF} and of the GMC partition function (2).

2.1 Discretized massive continuum GFF

First let us consdier the Gaussian free field. We would like to approximate a distribution on functions $\phi : \mathbb{T}^2 \to \mathbb{R}$ which should have probability density proportional to

$$\exp\left(-\frac{1}{2}\int_{\mathbb{T}^2} \|\nabla\phi(x)\|^2 + m^2|\phi(x)|^2\,dx\right) = \exp\left(-\frac{1}{2}\int_{\mathbb{T}^2}\langle\phi, (-\Delta + m^2)\phi\rangle\right), \tag{3}$$

where $\Delta = \nabla^2$ is the Laplacian on $\mathbb{T}^2$ (note that it is negative semidefinite).

Typically on a graph, $G = (V, E)$ the GFF is defined as a random function $\phi : V \to \mathbb{R}$ with probability density proportional to

$$\exp\left(-\frac{1}{2}\langle\phi, (-\Delta + m^2)\phi\rangle\right) = \exp\left(-\frac{1}{2}\sum_{x\sim y}(\phi_x - \phi_y)^2 - \frac{1}{2}m^2\sum_x \phi_x^2\right), \tag{4}$$

where $\Delta = \nabla^2$ is the graph Laplacian. However, each edge of our graph Ω_ε has length ε, so we should use the following difference operator:

$$(\nabla_\varepsilon\phi)(x, y) = \frac{\phi_x - \phi_y}{\varepsilon}.$$

Therefore we obtain the following Laplacian

$$\langle\phi, -\Delta_\varepsilon\phi\rangle = \sum_{x,y}\varepsilon^{-2}(\phi_x - \phi_y)^2.$$

Also, note that we want to replace $\int_{\mathbb{T}^2}$ by a normalized sum over Ω_ε so that the total mass is 1, i.e. we should take

$$\int_{\Omega_\varepsilon} = \varepsilon^2\sum_{x\in\Omega_\varepsilon}.$$

Therefore our GFF $\nu_m^{\mathrm{GFF}_\varepsilon}$ is a measure on functions $\phi : \Omega_\varepsilon \to \mathbb{R}$ with density proportional to

$$\exp\left(-\frac{1}{2}\sum_{x\sim y}(\phi_x - \phi_y)^2 - \frac{1}{2}m^2\varepsilon^2\sum_x \phi_x^2\right). \tag{5}$$

Notice how the only difference between (5) and (4) is that the mass is multiplied by ε.

By the way, the mass term is necessary to have a well-defined distribution in (4), since otherwise the measure would be shift invariant and thus would not be normalizable. There are some other options, like imposing boundary conditions, but I guess this would not be natural since we are working on a torus. If we did that instead in (4), then the variance of ϕ_x for some x in the bulk under the GFF measure would be

$$\frac{1}{2\pi}\log\frac{1}{\varepsilon} + O(1). \tag{6}$$

I won't go into the details of this, but this is roughly because the inverse of the Laplacian is the Green's function, which counts the expected number of returns of a random walk before hitting the boundary. In two dimensions on a box of side length n, this is of order $\log n$; the 2D random walk is just *barely* recurrent.

Now in (5), we are giving the measure a mass of εm in order to arrive at something which approximates a measure of the form (3). But since the typical value of ϕ_x is only of order

$$\sqrt{\log\frac{1}{\varepsilon}},$$

this tilt does almost nothing at any individual vertex $x \in \Omega_\varepsilon$, since the effect on one vertex is to change the tilt by the exponential of $m^2\varepsilon^2\log\frac{1}{\varepsilon} \to 0$. This means that the variance of ϕ_x under the measure $\nu_m^{\mathrm{GFF}_\varepsilon}$ with density given by (5) is still equal to the expression (6), but where the $O(1)$ term depends on m. When summing over all $x \in \Omega_\varepsilon$, the effect of the mass is noticeable, so it does change the overall distribution $\nu_m^{\mathrm{GFF}_\varepsilon}$.

2.2 Discretized GMC partition function and tilt

To define (2) in the discretized setting, we will consider the *Wick ordering*

$$:\exp\Big(\pm\sqrt{\beta}\phi_x\Big):_\varepsilon \coloneqq \varepsilon^{\beta/4\pi}\exp\Big(\pm\sqrt{\beta}\phi_x\Big).$$

The purpose of this normalization is so that the resulting discretized version of the partition function $\mathcal{Z}^\beta$ from (2) has mean of order 1. Specifically, let us define

$$\mathcal{Z}_\varepsilon^\beta(\phi) \coloneqq \int_{\Omega_\varepsilon} :\exp\Big(\sqrt{\beta}\phi_x\Big):_\varepsilon = \varepsilon^2\sum_{x\in\Omega_\varepsilon}\varepsilon^{\beta/4\pi}\exp\Big(\sqrt{\beta}\phi_x\Big). \tag{7}$$

Since under $\nu_m^{\mathrm{GFF}_\varepsilon}$ the individual point ϕ_x has variance $\frac{1}{2\pi}\log\frac{1}{\varepsilon}+O_m(1)$, we have

$$\mathbb{E}\big[\mathcal{Z}_\varepsilon^\beta(\Phi^{\mathrm{GFF}_\varepsilon})\big] = \varepsilon^{\beta/4\pi}\exp\left(\frac{1}{2}\beta^2\left(\frac{1}{2\pi}\log\frac{1}{\varepsilon}+O_m(1)\right)\right) = O_m(1),$$

where $\Phi^{\mathrm{GFF}_\varepsilon}\sim\nu_m^{\mathrm{GFF}_\varepsilon}$. Now we can define the ε-discretized Liouville model as follows:

$$d\nu^{\mathrm{Lv}_\varepsilon}(\phi) \propto e^{-\lambda\mathcal{Z}_\varepsilon^\beta(\phi)}\,d\nu_m^{\mathrm{GFF}_\varepsilon}(\phi), \tag{8}$$

where $\nu_m^{\mathrm{GFF}_\varepsilon}$ has density given by (5) and $\mathcal{Z}_\varepsilon^\beta$ is defined in (7) above. Of course, we can define the discretized sinh-Gordon model $\nu^{\mathrm{ShG}_\varepsilon}$ in much the same way.

2.3 Slightly more formal statement of results

Here is just one piece of the main theorem of [5]. It says that the Liouville and sinh-Gordon models may be coupled with the base GFF measure in such a way that the difference is smooth in a certain sense.

Theorem 2 (Still informal)**.** *For* $\mathcal{E}=\mathrm{Lv}$ *or* ShG*, we may write*

$$\Phi^{\mathcal{E}_\varepsilon} = \Phi^{D_\varepsilon} + \Phi^{\mathrm{GFF}_\varepsilon},$$

where $\Phi^{\mathcal{E}_\varepsilon}\sim\nu^{\mathcal{E}_\varepsilon}$*,* $\Phi^{\mathrm{GFF}_\varepsilon}\sim\nu_m^{\mathrm{GFF}_\varepsilon}$*, and*

$$\sup_{\varepsilon>0}\mathbb{E}\Big[\big\|\Phi^{D_\varepsilon}\big\|_{H^1}^2\Big] < \infty.$$

Here the Sobolev norm $\|\cdot\|_{H^1}$ on the discretized space Ω_ε is defined in terms of the Fourier transform, and I won't say much more about this. It turns out that the discretized variables mentioned in the theorem converge in a suitable sense to continuum versions in the corresponding Sobolev spaces as $\varepsilon\to0$, and the smoothness of the difference field, since it is uniform in ε, applies to the limit as well; thus the limiting difference field has one derivative. Since the GFF itself is only in $H^{-\kappa}$ for any $\kappa>0$, the difference is much smoother than either the base or tilted measure.

3 Polchinski dynamics

We would like to understand the measure $\nu^{\mathrm{Lv}_\varepsilon}$ defined by (8). As a start (which will also be a finish), let us try to understand the partition function of this model, i.e.

$$\mathbb{E}\Big[e^{-\lambda\mathcal{Z}_\varepsilon^\beta(\Phi^{\mathrm{GFF}_\varepsilon})}\Big]. \tag{9}$$

To understand this, we will decompose $\Phi^{\mathrm{GFF}_\varepsilon}$ into a sum of independent Gaussians which each act at a different scale, roughly speaking. This derives from a decomposition of the covariance matrix of $\nu_m^{\mathrm{GFF}_\varepsilon}$.

3.1 Covariance decomposition

First notice that for any Gaussian random variable Φ with covariance matrix C, we may write

$$\Phi = \int_0^\infty \dot{C}_s^{1/2}\, dW_s, \tag{10}$$

if W is a standard Brownian motion and

$$C = \int_0^\infty \dot{C}_s\, ds \tag{11}$$

is a deterministic decomposition of the covariance matrix into PSD parts $\dot{C}_s$. This is because the increments dW_s are independent, so the covariance matrix of the integral in (10) is the integral in (11).

Given a covariance decomposition as above, we write

$$C_t = \int_0^t \dot{C}_s\, ds,$$

so that $\Phi = \Phi_{<t} + \Phi_{>t}$, where $\Phi_{<t}$ has covariance matrix C_t, $\Phi_{>t}$ has covariance matrix $C - C_t$, and the two variables are independent. These are obtained simply by integrating over the correct intervals. So we may evaluate expectations of functions of Φ iteratively:

$$\mathbb{E}[F(\Phi)] = \mathbb{E}[\mathbb{E}[F(\Phi_{<t} + \Phi_{>t})|\Phi_{>t}]], \tag{12}$$

effectively integrating out the "small scales" first and then integrating out the large scales.

In our setting, the covariance matrix of $\Phi^{\mathrm{GFF}_\varepsilon}$ is

$$\left(-\varepsilon^2\Delta_\varepsilon + \varepsilon^2 m^2\right)^{-1} = \varepsilon^{-2}\left(-\Delta_\varepsilon + m^2\right)^{-1}.$$

So finding a covariance decomposition of this is the same as finding one for $\left(-\Delta_\varepsilon + m^2\right)^{-1}$. There are many choices here, but the one used by [5] is the Pauli-Villars decomposition

$$C_t = \varepsilon^{-2}\left(-\Delta_\varepsilon + m^2 + \frac{1}{t}\right)^{-1}.$$

For small t, this effectively encodes a GFF with a larger mass, which has shorter-range correlations. Indeed, fro t on the order of ε^2, this becomes the same as (4), which has exponential decay of correlations. Thus $C - C_t$ must encode the longer-range correlations in the actual GFF for such values of t.

3.2 Renormalized potentials and the Polchinski PDE

Let us return to calculating the partition function of $\nu^{\mathrm{Lv}_\varepsilon}$, given by (9). Let us set $v_0(\phi) = \lambda \mathcal{Z}_\varepsilon^\beta(\phi)$, so that we wish to calculate

$$\mathbb{E}_C\left[e^{-v_0(\zeta)}\right],$$

where $C = \varepsilon^{-2}\left(-\Delta_\varepsilon + m^2\right)^{-1}$. Here and in the future we write $\mathbb{E}_C$ for the expectation with respect to ζ, which is a Gaussian with covariance matrix C. By (12), if we define

$$e^{-v_t(\phi)} = \mathbb{E}_{C_t}\left[e^{-v_0(\phi+\zeta)}\right],$$

then the partition function (9) can also be written as

$$\mathbb{E}_{C-C_t}\left[e^{-v_t(\zeta)}\right].$$

This might not seem like a huge advantage, since we have essentially just pushed the problem to a different place: now we are integrating out a simpler Gaussian but we have a more complicated function. But the renormalized potential v_t turns out to satisfy the *Polchinski equation*

$$\partial_t v_t = \frac{1}{2}\Delta_{\dot{C}_t} v_t - \frac{1}{2}\|\nabla v_t\|^2_{\dot{C}_t}. \tag{13}$$

Here, for any covariance matrix C we have defined

$$\Delta_C = \sum_{x,y} C_{xy} \frac{\partial^2}{\partial \phi_x \partial \phi_y} \qquad \text{and} \qquad \langle u, v \rangle_C = \sum_{x,y} C_{xy} u(\phi_x) v(\phi_y).$$

Note that this Laplacian is not a graph Laplacian or space Laplacian of any kind, it acts on *functions of our random variable*.

To see why (13) holds, let us write $F_t(\phi) = e^{-v_t(\phi)}$ and note that $F_t(\phi) = \mathbb{E}_{C_t}[F_0(\phi + \zeta)]$. We will show that

$$\frac{\partial}{\partial t} F_t = \frac{1}{2} \Delta_{\dot{C}_t} F_t, \tag{14}$$

which will imply that $v_t = -\log F_t$ satisfies

$$\frac{\partial}{\partial t} v_t = -\frac{\frac{\partial}{\partial t} F_t}{F_t} = -\frac{\Delta_{\dot{C}_t} F_t}{2F_t} = -\frac{1}{2} e^{v_t} \Delta_{\dot{C}_t} e^{-v_t} = \frac{1}{2} \Delta_{\dot{C}_t} v_t - \frac{1}{2} \|\nabla v_t\|^2_{\dot{C}_t}.$$

To see (14), note that for small $\delta > 0$ we have

$$\begin{aligned}
\frac{\partial}{\partial t} F_t &\approx \frac{1}{\delta} \left(\mathbb{E}_{C_{t+\delta}}[F_0(\phi + \zeta_{t+\delta})] - \mathbb{E}_{C_t}[F_0(\phi + \zeta_t)] \right) \\
&= \frac{1}{\delta} \cdot \mathbb{E}_{C_{t+\delta} - C_t}[\mathbb{E}_{C_t}[F_0(\phi + \zeta_t + \zeta_\delta)] - \mathbb{E}_{C_t}[F_0(\phi + \zeta_t)]] \\
&\approx \frac{1}{\delta} \mathbb{E}_{\delta \dot{C}_t}[F_t(\phi + \zeta_\delta) - F_t(\phi)] \\
&\approx \frac{1}{\delta} \mathbb{E}_{\delta \dot{C}_t} \left[\langle \zeta_\delta, \nabla F_t(\phi) \rangle + \frac{1}{2} \sum_{x,y} \zeta^\delta_x \zeta^\delta_y \frac{\partial^2 F_t}{\partial \phi_x \partial \phi_y}(\phi) \right] \\
&= 0 + \Delta_{\dot{C}_t} F_t(\phi).
\end{aligned}$$

The Polchinski equation (13) can be used with some PDE techniques, I believe, to understand the *smoothness* of v_t. Intuitively, (13) is sort of like a heat equation, meaning that the solution v_t gets smoother as time increases, and so ∇v_t is tiny for large t. This will be relevant in the next section.

3.3 Polchinski semigroup and BSDE for renormalized measure

Let us define the Polchinski semigroup $P_{s,t}$ via

$$P_{s,t} F(\phi) = e^{v_t(\phi)} \mathbb{E}_{C_t - C_s} \left[e^{-v_s(\phi + \zeta)} F(\phi + \zeta) \right].$$

In other words, $P_{s,t}$ averages out the scales between s and t. Note that ϕ should be thought of as the value at time t, so that we start by knowing the large scales and we want to find out what the function looks like averaged over the small scales, given the large scales equal ϕ. Also, this is actually a semigroup, since for $s \le r \le t$, we have

$$\begin{aligned}
P_{r,t} P_{s,r} F(\phi) &= e^{v_t(\phi)} \mathbb{E}_{C_t - C_r} \left[e^{-v_r(\phi + \zeta_{rt})} e^{v_r(\phi + \zeta_{rt})} \mathbb{E}_{C_r - C_s} [e^{-v_s(\phi + \zeta_{rt} + \zeta_{sr})} F(\phi + \zeta_{rt} + \zeta_{sr})] \right] \\
&= e^{v_t(\phi)} \mathbb{E}_{C_t - C_s} \left[e^{-v_s(\phi + \zeta)} F(\phi + \zeta) \right] = P_{s,t} F(\phi).
\end{aligned}$$

We may also define the renormalized measure ν_t via

$$\mathbb{E}_{\nu_t}[F] = P_{t,\infty} F(0) = e^{v_\infty(0)} \mathbb{E}_{C_\infty - C_t} \left[e^{-v_t(\zeta)} F(\zeta) \right].$$

In other words, this averages over all scales larger than t, so the measure ν_t is something which can only see these larger scales, and replaces all small scales by their averaged versions. Note that for $s \le t$, we have

$$\mathbb{E}_{\nu_s}[F] = P_{s,\infty} F(0) = P_{t,\infty} P_{s,t} F(0) = \mathbb{E}_{\nu_t}[P_{s,t} F],$$

so the semigroup $P_{s,t}F$ takes the measure ν_t and gives ν_s. The true measure $\nu^{\mathrm{Lv}_\varepsilon}$ equals ν_0, and this is what we would like to understand.

How can we sample from the measure ν_0, or the measure ν_s in general? If we had a sample from ν_t for $t > s$, then we could try to "add back in" the smaller scale behavior to get a sample from ν_s. It turns out we can do this with the following backwards SDE:

$$d\Phi_t = -\dot{C}_t \nabla v_t(\Phi_t)\, dt + \dot{C}_t^{1/2}\, dW_t. \tag{15}$$

This may seem reminiscent of Langevin dynamics, which does stochastic gradient descent to sample from a fixed measure of the form e^{-v}. Here, though, the dynamics depends on time.

The main result is that if $\Phi_t \sim \nu_t$ and Φ_r satisfies the equation (15) for $s \le r \le t$, then $\Phi_s \sim \nu_s$. This is implied by the fact that if $s \le t$, we have

$$P_{s,t}F(\phi) = \mathbb{E}_{\Phi_t = \phi}[F(\Phi_s)] \tag{16}$$

for any bounded function F. To check (16), note that for any function $f(t,\phi)$, Itô's lemma gives

$$\begin{aligned} df(t,\Phi_t) &= \frac{\partial f}{\partial t}(t,\Phi_t)\, dt + \left\langle \nabla f(t,\Phi_t), \dot{C}_t \nabla v_t(\Phi_t)\, dt + \dot{C}_t^{1/2}\, dW_t \right\rangle - \frac{1}{2}\Delta_{\dot{C}_t} f(t,\Phi_t)\, dt \\ &= \left(\frac{\partial f}{\partial t}(t,\Phi_t) - \frac{1}{2}\Delta_{\dot{C}_t} f(t,\Phi_t) + \langle \nabla f(t,\Phi_t), \nabla v_t(\Phi_t)\rangle_{\dot{C}_t} \right) dt + \langle \nabla f(t,\Phi_t), \dot{C}_t^{1/2}\, dW_t\rangle. \end{aligned}$$

If we can show that the parenthesized expression above is zero for $f(t,\phi) = P_{s,t}F(\phi)$ then (16) will follow, since then

$$\mathbb{E}_{\Phi_t=\phi}[F(\Phi_s)] = \mathbb{E}_{\Phi_t=\phi}[P_{s,s}F(\Phi_s)] = P_{s,t}F(\Phi_t = \phi)$$

as required. This martingale condition follows from a calculation similar to the one used to prove (14) above.

3.4 The coupling

The result follows from coupling the SDE (15) with the simpler one

$$d\Phi_t = \dot{C}_t^{1/2}\, dW_t, \tag{17}$$

both with $\Phi_\infty = 0$, using the same driving noise. The result of (17) is that $\Phi_0^{\mathrm{GFF}_\varepsilon} \sim \nu_m^{\mathrm{GFF}_\varepsilon}$ whereas the result of (15) yields $\Phi_0^{\mathrm{Lv}_\varepsilon} \sim \nu^{\mathrm{Lv}_\varepsilon}$. The smoothness of the difference follows from bounds on ∇v_t. As mentioned, ∇v_t should be tiny for large enough t, and so the two processes should stay close until we get to the small scales. Ideally, they stay close enough that whatever happens at the small scales can't make the difference too rough. The tiny scales ($t \ll \varepsilon^2$) should be inconsequential by exponential decay of correlations, and the small but not tiny scales should already be smoothed out enough since the heat equation behavior of the Polchinski PDE hopefully smooths things out quickly.

References

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