Razborov's Triangle:

Flag algebras

$\rightsquigarrow$ certain functions of densities are ≥ 0

Mantel's thm: if $t=0$, $e \le \frac{1}{2}$.

$$e^2 \le \int_0^1 \left(\int_0^1 W(x,y)\,dy\right)^2 dx = \cancel{\int_0^1\int_0^1\int_0^1 W(x,y)\,W(x,z)\,dx\,dy\,dz}$$

symmetrize this since W = sym.

$$= \frac{1}{2}\int_0^1\int_0^1 W(x,y)\Big(\underbrace{\int_0^1 W(x,z)\,dz + \int_0^1 W(y,z)\,dz}_{\le 1 \text{ a.s. in } W(x,y) \text{ by triangle-freeness.}}\Big)\,dx\,dy$$

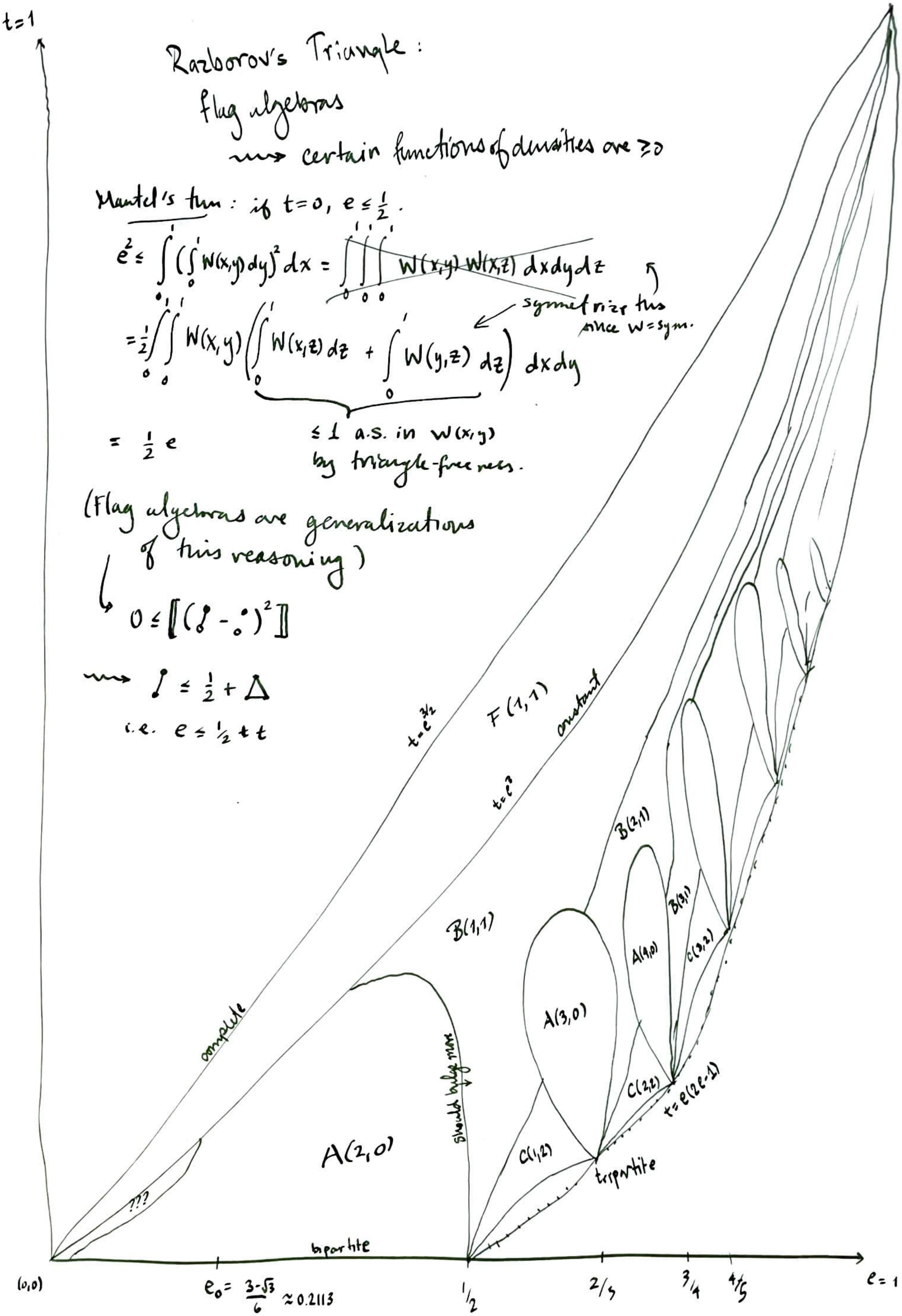
$$= \frac{1}{2}e$$

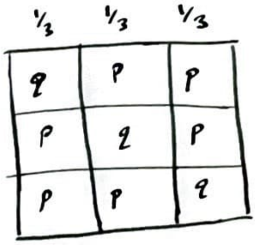
(Flag algebras are generalizations of this reasoning)

$\hookrightarrow$ $0 \le [\![(\overset{\bullet}{\underset{\bullet}{|}} - \overset{\bullet}{\circ})^2]\!]$

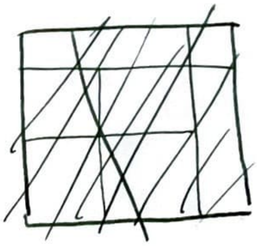
$\rightsquigarrow$ $\overset{\bullet}{\underset{\bullet}{|}} \le \frac{1}{2} + \Delta$

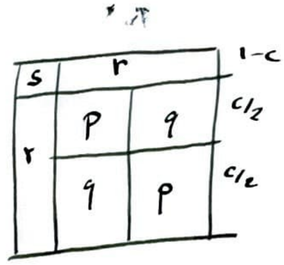
i.e. $e \le \frac{1}{2} + t$



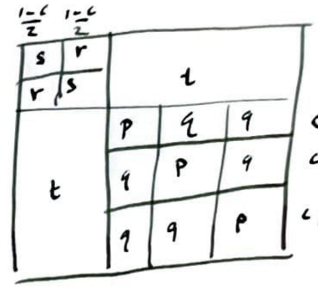


A(3,0)





B(2,1)



C(3,2)

(F(1,1) more complete, B(1,1) more bipartite).

LDP: the graphs which are typical with edge density e & trid. t,

maximize $H(W) = \iint_{00}^{11} H(W(x,y))\,dx\,dy$ subject to constraints.

in 2017, Kenyon, Radin, Ren, Sadun conjectured the phase diagram inside Razborov's triangle, primarily based on numerics.

$\rightsquigarrow$ this has been proven in many cases, still open in many others.

in 2023, Neeman, Radin & Sadun found a very tiny region with a counterexample, when $e < e_0 \approx 0.2113$ and t is just below e^3.

Their example beats the (2,0) graphon and has (2,1) symmetry.

Last week, DiCarlo & Sadun investigated this region of the phase diagram more.

Their work is purely numerical, and its kind of not clear what exactly they conclude... kind of not clear what not

there could be infinitely many different phases as $e,t \to 0$. $e,t \to 0$ $\to 0$

(they also don't seem to prove that the optimizer is tripodal in any region, right? but they do claim that "typical graphs turned out to be tripodal" so idk)

(they find the optimum, assuming tripodality though) (and see how far some tripodal beats bipodal).

Notes on flag algebra: $\mathcal{A}^\bullet$ = algebra of formal linear combinations of rooted graphs,

$t_W^\bullet(a)$ = density of a in W if root is chosen in $[0,1]$ u.a.r. ($t_W^\bullet$ is random).

$\rightsquigarrow$ product on $\mathcal{A}^\bullet$ chosen to make each $t_W^\bullet$ a homomorphism (a.s.)

$[\![a]\!] \in \mathcal{A}$ such that $\mathbb{E}_\bullet[t_W^\bullet(a)] = t_W([\![a]\!])$ $\rightsquigarrow$ standard hom densities.

formal lin. comb of graphs $\qquad t_W([\![a]\!])$

How to find the counterexample:

- if $W \approx e$ pointwise, then taylor expand H to obtain

$$H(W) = H(e) + \tfrac{1}{2} H''(e) \|W-e\|_{L^2}^2 + O(\|W-e\|_{L^2}^{4 \text{ or } 3})$$

(odd terms disappear via averaging, since $e(W)=e$)

so: need to instead have it differ greatly from e on a small region.
(of order δ^2 in area, if the overall L_2 dist is δ)
(so that the $O(\|W-e\|^3)$ term still is dominated).

this suggests:

$$\begin{array}{c|c|c|}
1-c & e+O(c) & e+O(c^2) \\
\hline
c & g_0 & e+O(c) \\
\hline
 & c & 1-c
\end{array}$$

- for $e \geq \frac{1}{2}$, taking g_0 to be constant is apparently optimal.
- next simplest possibility is for g_0 to be $(2,0)$ (i.e. symmetric bipodal).

How to conjecture the Phase Diagram:

just assume that the optimizer is multipodal with ≤ 16 podes, and uh... do numerical optimization for the entropy of a graphon, in terms of ≤ 1000 parameters.

Where phase diagram has been proven:

- just above E-R curve $e^3 = t$
- just below ER curve when $e \geq 1/2$
- near the line segment $e = 1/2$, $0 < t < 1/8$
- just below the top boundary $t = e^{3/2}$
- just above each of the "scallops", including the bipartite bottom segment.

To review: 1: Question: what is the structure of graphs with a set number of edges & triangles? and how many such graphs are there?
(answer: $\frac{\log \#}{n^2} \longrightarrow \max_W \{H(W) : e(W)=e, t(W)=t\}$
& they look like argmax).

2. what is a graphon? (introduce subgraph densities)

$\rightsquigarrow$ Large deviations principle for $G(n, 1/2)$ in terms of graphon entropy function.