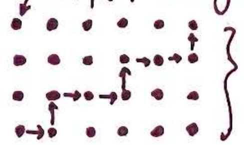
Theorem:

$$\frac{G(m,n)-(\sqrt{m}+\sqrt{n})^2}{(mn)^{-1/6}(\sqrt{m}+\sqrt{n})^{4/3}} \xrightarrow{d} TW_2$$

- $G(m,n)$ = last passage time from $(0,0)$ to (m,n) in Exp(1) LPP on the 1^{st} quadrant $\mathbb{N}^2$.

i.e.

Exp(1) weights on vertices



$\}$ weight of path $= \sum$ weights of vertices

$G(m,n)$ = weight of maximal directed path $(0,0) \to (m,n)$

- TW_2 = limiting distribution of highest eigenvalue in GUE.

  = law of maximal point in the Airy process, which is the distributional limit of the rescaled eigenvalues of GUE at the edge of the semicircle

Notes:
- convergence holds as $m,n\to\infty$ with $0<\liminf \frac{m_k}{n_k} \le \limsup \frac{m_k}{n_k} <\infty$.
- The limit shape (for CGP) (LLN) result can be proven separately, and it involves some technical details and an optimization problem, and the Legendre transform. There is also a limit shape for MCGP

Main Ingredients

① Formula:

$$\mathbb{P}[G(m,n)\le t] = \frac{1}{n!}\prod_{j=0}^{n-1}\frac{1}{j!(m-n+j)!}\int_0^t\cdots\int_0^t \prod_{1\le i<j\le n}(x_i-x_j)^2\prod_{j=1}^n x_j^{m-n}e^{-x_j}\,dx_1\cdots dx_j$$

② (Fredholm) Determinantal formula in terms of Laguerre kernel: $\mathbb{P}[G(m,n)\le t] = \det\left(I-(L_n^{m,n})_{[t,\infty)}\right)$

③ Airy Asymptotics for Laguerre kernel: (shifted & scaled $L_n(s,t)$) $\longrightarrow A(s,t)$, the Airy kernel.

① a: connect Geom(p) LPP (i.e. MCGP) to generalized permutations ~~via the RSK algorithm~~ (via the matrix representation of a generalized permutation)
$\to$ longest increasing subseq of gen. perm $\leftrightarrow$ passage time matrix $= G_p(m,n)$.

b: connect generalized permutations to SSYT via the RSK correspondence, and enumerate SSYT by counting Gelfand-Tsetlin patterns.

c: get exact formula for $\mathbb{P}[G_p(m,n)\le t]$, and get Exp(1) as a limit of $p\cdot$Geom(p) as $p\searrow 0$.

Note: This is the same formula found when studying Wishart matrices X^*X, and this shows that $G(m,n) \stackrel{d}{=}$ maximal eigenvalue of Wishart matrix when $X = m\times n$ array of iid $N_{\mathbb{C}}(0,1)$

② a: Laguerre polynomials $\ell_n^\alpha(x) = \left(\frac{n!}{(n+\alpha)!}\right)^{1/2}\sum_{k=0}^n \frac{(-1)^{n+k}}{k!}\binom{n+\alpha}{k+\alpha}x^k$ are the orthogonal polynomials associated with the weight function $e^{-x}x^\alpha \mathbb{1}_{[0,\infty)}(x)$.

b: $L_n^\alpha(x,y) = \sqrt{n(n+\alpha)}\, x^{\alpha/2}y^{\alpha/2}e^{-x/2}e^{-y/2}\times\begin{cases}\frac{\ell_n^\alpha(x)\ell_{n-1}^\alpha(y)-\ell_{n-1}^\alpha(x)\ell_n^\alpha(y)}{x-y} & \text{if } x\ne y,\\ (\ell_n^\alpha)'(x)\ell_{n-1}^\alpha(y) - (\ell_{n-1}^\alpha)'(x)\ell_n^\alpha(y) & \text{if } x=y\end{cases}$
is the Laguerre kernel.

c: by general orthogonal polynomial business, since $e^{-x}x^\alpha$ shows up in formula ①, it turns out to be the same as the Fredholm determinantal formula

③ a: use ~~Cauchy~~ contour integral formula for ℓ_n^α (via power series + residue theorem) to write Laguerre kernel in terms of a contour integral (actually many)

b: Apply the saddle point analysis to the contour integral which is used to define L_n^α, and after expanding near the saddle point, find that the integral which defines the Airy function appears as the asymptotic.

c: do a bunch of analysis to make sure all the convergences work out.

Note: also need a couple of bounds on L_n^α to make sure that the convergence of one kernel to another also implies that $\mathbb{P}[\text{shifted scaled } G(m,n)\le t]\longrightarrow \mathbb{P}[TW_2\le t]$