Combining Biquandle Cohomological and State-Sum Polynomial Knot Invariants

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Overview

Consider two separate knot invariants: Boltzman sums of 2-cocycle weights arising from biquandle cohomology and biquandle brackets involving state-sum polynomial splitting coefficients. We determine that constructions of certain enhanced knot invariants for biquandle-colored knot diagrams are in fact factorable into these two separate knot invariants.

Quandles & Biquandles

Definition A *biquandle* is a set X with two binary operations $\triangleright, \trianglerighteq$ such that $\forall x, y, z \in X$,

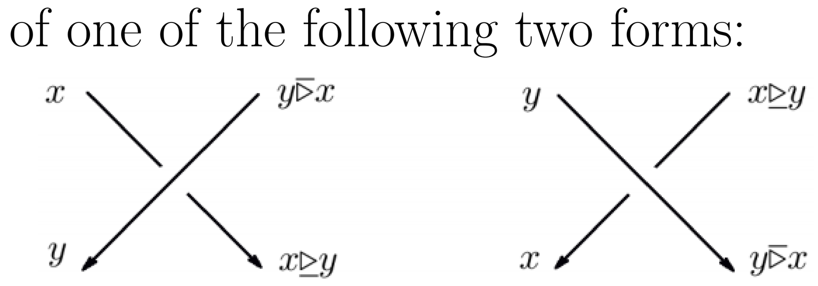
- $x \trianglerighteq x = x \triangleright x$
- The maps $\alpha_y(x) = x \triangleright y, \beta_y(x) = x \trianglerighteq y$, and $S(x,y) = (y \triangleright x, x \trianglerighteq y)$ are invertible.
- The exchange laws are satisfied:

$$(x \trianglerighteq y) \trianglerighteq (z \trianglerighteq y) = (x \trianglerighteq z) \trianglerighteq (y \triangleright z)$$
$$(x \trianglerighteq y) \triangleright (z \trianglerighteq y) = (x \triangleright z) \trianglerighteq (y \triangleright z)$$
$$(x \triangleright y) \triangleright (z \triangleright y) = (x \triangleright z) \triangleright (y \trianglerighteq z).$$

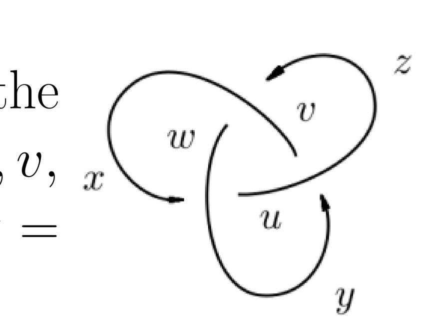
If $x \triangleright y = x$ for all $x, y \in X$, then X is called a *quandle*.

Biquandle Coloring

Definition The *fundamental biquandle* of a link L, denoted $\mathcal{B}(L)$, is the biquandle generated by the semiarcs of any diagram for L and the crossing relations. This means if a crossing occurs in the diagram then it must be of one of the following two forms:



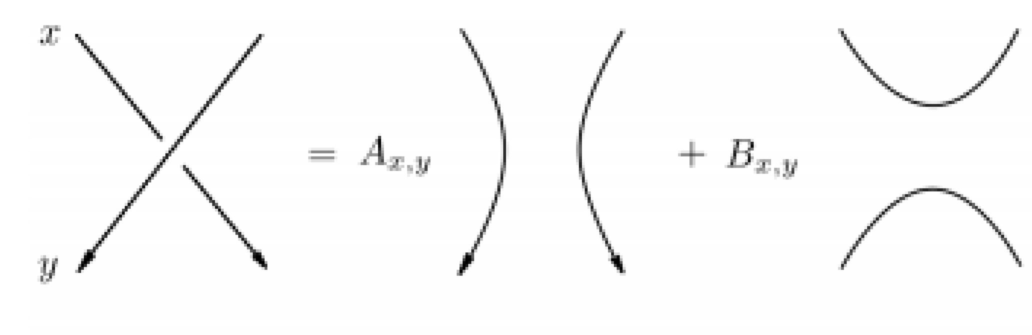
For instance, the fundamental biquandle $\mathcal{B}(3_1)$ of the trefoil knot is generated by the elements x, y, z, u, v, and w, which satisfy $x \trianglerighteq y = u, y \triangleright x = w, y \trianglerighteq z = v, z \triangleright y = u, z \trianglerighteq x = w$, and $x \triangleright z = v$.

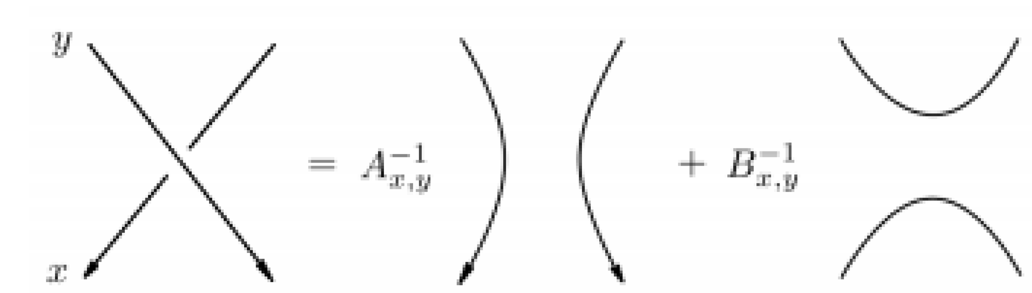


Definition a *biquandle coloring* of a link L by a biquandle X is a biquandle homomorphism from $\mathcal{B}(L)$ into X. A biquandle coloring of a link L can be seen as an extension of some biquandle coloring of a diagram D for L, wherein each semiarc of D is assigned an element (its "color") from X so that the crossing relations are satisfied.

Biquandle Brackets

Definition Let X be a finite biquandle, and let R be a commutative ring with identity, a *biquandle bracket* is a pair of maps $A, B : X^2 \to R^x$ determined by the skein relations





which define appropriate factors for writhe and unions of unknots to be invariant under the Reidemeister moves. A polynomial knot invariant is obtained by summing over all the states obtained via the possible splittings using splitting weights found in the biquandle bracket.

Definition Given a finite biquandle $X = \{x_1, ..., x_n\}$, a biquandle bracket can be represented by a pair of $n \times n$ matrices A, B with $A_{j,k} = A(j,k)$ and $B_{j,k} = B(j,k)$. For convenience, we write these as a single $n \times 2n$ block matrix. We call this a *biquandle bracket presentation matrix.*

2-Cocycle Invariants

Definition A map $\phi : X^2 \to G$, where G is an abelian group, is a *2-cocycle* if $\forall x, y, z \in X$:

- $\phi(x,x) = 1$
- $\phi(y,z)\phi(x,y)\phi(x \trianglerighteq y, z \triangleright y) = \phi(x,z)\phi(y \triangleright x, z \triangleright x)\phi(x \trianglerighteq z, y \trianglerighteq z)$

If X is a quandle, the second constraint for the 2-cocycle reduces to:

$$\phi(x,y) \cdot \phi(x \trianglerighteq y, z) = \phi(x,z) \cdot \phi(x \trianglerighteq z, y \trianglerighteq z)$$

The operation matrix M for a 2-cocycle is described by $M_{ij} = \phi(x_i, x_j)$ for $i, j = 1, 2, ..., n$ and $x_i, x_j \in X$.

Given a knot diagram D with a set of biquandle colorings $\mathcal{C} = \mathrm{Hom}(\mathcal{B}(D), X)$ and crossing set $\mathcal{T}$, the biquandle 2-cocycle enchancement is written multiplicatively as follows:

$$\Phi^{\phi}_X(D) = \Sigma_{C \in \mathcal{C}} \Pi_{\tau \in T} \phi(x_\tau, y_\tau)^{\epsilon(\tau)}$$

where ϵ is the function returning the sign of the crossing, ± 1.

Main Result

Let P be a biquandle bracket presentation matrix. If $P = Q \otimes M$, where Q is also a biquandle bracket presentation matrix, then M is an operation matrix for a 2-cocycle (up to a scalar multiple). In essence, any knot invariant that can be decomposed this way is actually the product of two separate knot invariants.

Examples

Splitting Yang's Enhanced Kauffman Bracket in [3]:
Consider a bicolored knot diagram, i.e. a diagram colored by $X = \mathbb{Z}_2$, where we define $x \trianglerighteq y = x \triangleright y = 1 - x \quad \forall x, y \in X$ (this biquandle "flips" the left argument in any operation).
Let $a, b, n, e, w \in R^x$

$$\left(\begin{array}{cc|cc} na & ea & nb & eb \\ wa & na & wb & nb \end{array}\right) = \left(\begin{array}{c|c} na & nb \end{array}\right) \otimes \begin{pmatrix} 1 & e/n \\ w/n & 1 \end{pmatrix}$$

Consider a tricolored knot diagram, i.e. a diagram colored by a Takasaki quandle $X = \mathbb{Z}_3$, with $x \trianglerighteq y = 2y - x$ and $x \triangleright y = x$.
Let $a, b, c, n, s \in R^x$ with $n^3 = c^3 = s^3$

$$\left(\begin{array}{ccc|ccc} ca & na & sa & cb & nb & sb \\ sa & ca & na & sb & cb & nb \\ na & sa & ca & nb & sb & cb \end{array}\right) = \left(\begin{array}{c|c} ca & cb \end{array}\right) \otimes \begin{pmatrix} 1 & n/c & s/c \\ s/c & 1 & n/c \\ n/c & s/c & 1 \end{pmatrix}$$

References

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[2] Nelson, Sam, Michael E. Orrison, and Veronica Rivera. "Quantum enhancements and biquandle brackets." Journal of Knot Theory and Its Ramifications 26.05 (2017): 1750034.
[3] Yang, Zhiqing. "Enhanced Kauffman bracket." arXiv preprint arXiv:1702.03391 (2017).

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